

## NON-SEMISIMPLE MACDONALD POLYNOMIALS

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<sup>†</sup> November 1, 2008 Partially supported by NSF grant DMS-0456445.

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**Basic notations.**

1)  $R = \{\alpha\} \subset \tilde{R} = \{[\alpha, \nu_\alpha j], j \in \mathbb{Z}\}$  : root systems in  $R^n$  with the form  $(, )$  and  $R^{n+1}$ ;  $\nu_\alpha = (\alpha, \alpha)/2$ ,  $\nu_{\text{sht}} = 1$ .

2)  $W \subset \widetilde{W} = \langle s_i, 0 \leq i \leq n \rangle = W \ltimes Q \subset \widehat{W} = W \ltimes P$  for the root and weight lattices  $Q, P$ ;  $l(\widehat{w}), l_\nu(\widehat{w})$  are the length and partial length;  $B$  is a lattice between  $Q$  and  $P$ .

3)  $\Pi = P/Q = \{\pi_r, r \in O\}$ ,  $O$  is the orbit of  $\alpha_0$  in the affine Dynkin diagram  $\tilde{\Gamma}$ ;  $\alpha_0 = [-\vartheta, 1]$  for the maximal short root  $\vartheta$ ;  $O' = O \setminus \alpha_0$ ;  $\widehat{W}^b = W \ltimes B = \widetilde{W} \rtimes \Pi^b$  for the image  $\Pi^b$  of  $B$  in  $\Pi$ .

4)  $\lambda(\widehat{w}) = \tilde{R}_+ \cap \widehat{w}^{-1}(\tilde{R}_-)$ ,  $\widehat{w} \in \widehat{W}$ ; for reduced  $\widehat{w} = \pi_r s_{i_1} \cdots s_{i_2} s_{i_1}$ ,  $\lambda(\widehat{w}) = \{\tilde{\alpha}^l = \tilde{w}^{-1} s_{i_l}(\alpha_{i_l}), \dots, \tilde{\alpha}^2 = s_{i_1}(\alpha_{i_2}), \tilde{\alpha}^1 = \alpha_{i_1}\}$ ;  $\tilde{w} = \pi_r^{-1} \widehat{w}$ .

5)  $(wb)((z)) = w(b+z)$ ,  $([z, l], z' + d) = (z, z') + l$  for  $w \in W$ ,  $b \in P$ ,  $z \in \mathbb{C}^n$ ;  $(\widehat{w}([z, l]), \widehat{w}((z')) + d) = ([z, l], z' + d)$  for  $\widehat{w} \in \widehat{W}$ ;  $\mathfrak{C} = \{z \in \mathbb{R}^n, (z, \alpha_i) > 0 \text{ as } i > 0\}$ ,  $\mathfrak{C}^a = \bigcap_{i=0}^n \mathfrak{L}_{\alpha_i}$ , where  $\mathfrak{L}_{[\alpha, \nu_\alpha j]} = \{z \in \mathbb{R}^n, (z, \alpha) + j > 0\}$ .

6)  $\rho_\nu = \frac{1}{2} \sum_{\nu_\alpha = \nu} \alpha = \sum_{\nu_i = \nu} \omega_i$ , where  $\alpha \in R_+$ ,  $\omega_i$  are fundamental weights;  $(\rho_\nu)^\vee = \rho_\nu / \nu$ ,  $2\rho_k = \sum_{\alpha \in R_+} k_\alpha \alpha$ ;  $q_\alpha = q^{\nu_\alpha}$ ,  $t_\alpha = q^{k_\alpha}$ .

7)  $b = \pi_b u_b$  for  $b \in P$ ,  $u_b \in W$ , where  $u_b(b) = b_- \in P_-$  and  $\lambda(\pi_b) \cap R = \emptyset$ ;  $\pi_r = \pi_{\omega_r}$  for  $r \in O'$ ;  $u_b \pi_b = b_-$ ,  $b_+ = w_0(b_-) = \varsigma(-b_-)$ ;  $b_\# = b - u_b^{-1}(\rho_k)$ .

8)  $\mathcal{V} = \mathbb{Q}_{q,t}[X_b] = \mathbb{Q}_{q,t}[X_b, b \in B]$  : the polynomial representation over  $\mathbb{Q}_{q,t} = \mathbb{Q}[q^{\pm 1/m}, t^{\pm 1/2}]$  for  $(B, B) = (1/m)\mathbb{Z}$ ;  $\Phi_{\widehat{w}}$  and  $\Psi_{\widehat{w}}$  are  $X$  and  $Y$  intertwiners;  $\Psi_i = \tau_+(T_i) + \frac{t_i^{1/2} - t_i^{-1/2}}{Y_{\alpha_i} - 1}$ ,  $i \geq 0$ ,  $P_r = \tau_+(\pi_r)$ ,  $r \in O'$  for the DAHA-automorphism  $\tau_+$ .

9)  $\tilde{E}_b, E_b, P_b, \mathcal{E}_b = E_b/E_b(q^{-\rho_k})$  : non-semisimple, nonsymmetric, symmetric and spherical polynomials,  $b \in B$ ;  $\Psi_i^b$  is  $\Psi_i$  acting on  $E_b$ .

10)  $\tilde{V}_b \subset \mathcal{V}_b \subset \mathcal{V}(-b_\#)^\infty$  : spaces of generalized  $Y$ -eigenvectors in  $\mathcal{V}$ ;  $\tilde{V}_b$  may depend on the reduced decomposition of  $\pi_b(b \in B)$ ,  $\mathcal{V}_b = \lim \mathcal{V}_b^l$  does not depend on the decomposition;  $\tilde{\Psi}_i, \tilde{\Psi}_{\widehat{w}}$  are the generalized intertwiners defined in terms of  $\tilde{R}^0$ .

11)  $R^0$  is a root subsystem in  $\tilde{R}$ ,  $\widetilde{W}^0 = \langle s_\alpha, \alpha \in R^0 \rangle$ ; in the context of  $\mathcal{V}$ ,  $\tilde{R}^0 = \{\tilde{\alpha} = [\alpha, \nu_\alpha j] \in \tilde{R} \mid q^{\nu_\alpha j - (\alpha, \rho_k)} = q_\alpha^{j - (\alpha^\vee, \rho_k)} = 1\}$ ;  $\widehat{W}^b[\xi] = \{\widehat{w} \in \widehat{W}^b \mid q^{\widehat{w}(\xi)} = q^\xi\}$ ;  $\Pi_{\tilde{R}}$  is the product of affine exponents.

## 0. INTRODUCTION

The paper is mainly devoted to the irreducibility of the polynomial representation of the Double affine Hecke algebra, DAHA, for arbitrary reduced root systems and generic “central charge”  $q$ .

The technique of intertwiners from [C7] in a non-semisimple variant is the main tool. It is important for the decomposition of the polynomial representation in terms of irreducible DAHA modules and for its weight decomposition. We focus on the principal aspects of the technique of intertwiners and discuss only basic (and instructional) applications. Generally, it is more efficient to combine the intertwiners with other approaches, to be considered in further papers, including author’s (unpublished) construction of the Jantzen–type filtration.

There are several methods that can be used now in the study of the polynomial representation of DAHA and its degenerations. Certainly the localization functor (the KZ–monodromy) from [GGOR, VV2] must be mentioned, as well as the geometric methods of [VV3] and the parabolic induction from recent [BE]. The technique of intertwiners provides constructive (relatively elementary) tools for managing the irreducibility of the polynomial representation and its constituents for any  $q, t$  based on combinatorics of affine Weyl groups.

An important general objective of this technique is finding a counterpart of the classical theory of *highest vectors* for DAHA (and AHA), complementary to the geometric method of [KL1]. It involves difficult combinatorial problems and is known only for type  $A$  and in some cases of small ranks. However, the geometric DAHA methods are far from simple too and explicit theory of DAHA modules is needed in quite a few applications (see Section 0.4).

**Main constructions.** The *polynomial representation*, denoted by  $\mathcal{V}$  in the paper, is well known to be irreducible and semisimple for generic values of the DAHA-parameters  $q$  and  $t = q^k$ . It becomes reducible either when  $q$  is a root of unity or for generic  $q$  and special  $t$ .

We perform a complete analysis of the irreducibility and semisimplicity of  $\mathcal{V}$  for generic  $q$ . Another application is a construction of the canonical semisimple submodule in  $\mathcal{V}$  generalizing that of type  $A$  from [FJMM] (the symmetric variant) and [Ka] (the non-symmetric case).

We begin with a description of all *singular*  $t = q^k$  making (by definition) the radical of the evaluation pairing nonzero. The answer is

given in terms of the principal values (at  $t^{-\rho}$ ) of the nonsymmetric Macdonald polynomials [C5, C6, C11] for the weights sufficiently large to ensure the existence of these polynomials. The same answer can be obtained using a generalization of the method from [O1, O2] in the rational case based on the *shift-operator*. The latter is used to calculate the principal value of the  $t$ -discriminant  $D_Y$  applied to the  $t$ -discriminant  $D_X$  in  $\mathcal{V}$ , where  $X, Y$  are the generators of DAHA. The approach via the Macdonald polynomials has no rational counterpart.

The *evaluation pairing* is defined as follows:

$$\{E, F\} = E(Y^{-1})(F(X))(t^{-\rho}), \quad E, F \in \mathcal{V};$$

for instance,  $D_Y(D_X)(t^{-\rho}) = \{D_{X^{-1}}, D_X\}$ . In the simply-laced case, the radical *Rad* of this pairing is zero if and only if  $\mathcal{V}$  is irreducible. This equivalence becomes more subtle in the non-simply-laced case, as well as the formula for  $D_Y(D_X)(t^{-\rho})$ . The  $q, t$ -theory provides the best (and direct) method for calculating this formula, including managing the rational case through the limiting procedure from DAHA to its rational degeneration.

The main objects of this paper are the *chains of the intertwiners* and *non-semisimple Macdonald polynomials* (defined via such chains). We define a system of subspaces in  $\mathcal{V}$  with the Macdonald polynomials as top elements in a punctured neighborhood of a singular  $t$  and then extend this construction to singular  $t$ .

In contrast to the semisimple case, the related combinatorics of the affine root systems and affine Weyl groups becomes significantly more involved. For instance, the non-semisimple Macdonald polynomials may depend on the choice of the reduced decomposition of the corresponding elements in the affine Weyl groups, the *relative Bruhat ordering* is needed versus the usual one, and so on.

There are six Main Theorems in the paper, we will discuss its contents following these theorems beginning with those of combinatorial nature.

**0.1. Reduced decompositions.** A significant part of the paper is devoted to the combinatorics of affine root systems grouped around the reduced decompositions. The theory of reduced decompositions in affine (and non-affine) Weyl groups is far from being simple and complete.

The *affine exponents* (see below), generalizing the classical Coxeter exponents, demonstrate that there are many properties of the reduced decompositions we do not know.

Almost all facts of combinatorial nature we mention/prove in the paper are really needed here. Some, like Main Theorem 2.4, are expected to be used in other papers, say, for the classification of the semisimple representations and in the theory of Jantzen–type filtration.

Generally speaking, many (if not all) theorems established in [C12] and previous author’s papers (including the Macdonald conjectures) are corollaries of relatively few facts on affine root systems and affine Weyl groups. Extending the list of such basic facts is very important; any progress here can be readily translated to the DAHA and AHA theories. This paper demonstrates it.

We note that an abstract foundation of our approach is the notion of a pair of *compatible  $r$ -matrices* associated with an affine root system and its root subsystem ( $\tilde{R}^0 \subset \tilde{R}$ , see below). There is a connection with Kauffman’s axioms of *virtual links* in topology [GPV] (when extended to arbitrary affine root systems), although the relation is direct only for some representations of DAHA.

Let us briefly discuss the main theorems of the “combinatorial part” of this paper. We begin with an irreducible reduced root system  $R$  and its affine extension  $\tilde{R} = \{[\alpha, \frac{(\alpha, \alpha)}{2}j]\}$ , where  $\alpha \in R$ ,  $j \in \mathbb{Z}$ ;  $W$  is the Weyl group of  $R$ ,  $\widehat{W}$  is the extended affine Weyl group  $\widehat{W} = W \ltimes P$  defined for the weight lattice  $P$  of  $R$ .

*Main Theorem 2.1.* The key tool we use in the theory of the extended affine Weyl group  $\widehat{W}$  is the notion of the  $\lambda$ -set:  $\lambda(\widehat{w}) = \tilde{R}_+ \cap \widehat{w}^{-1}(-\tilde{R}_+)$  for  $\widehat{w} \in \widehat{W}$ , where  $\tilde{R}_+$  is the set of positive roots in  $\tilde{R}$ . It is well-known that  $\widehat{w}$  is uniquely determined by  $\lambda(\widehat{w})$ ; many properties of  $\widehat{w}$  and its reduced decompositions can be interpreted in terms of this set. We give an intrinsic description of the  $\lambda$ -sets, more generally, the  $\lambda$ -sequences, that are  $\lambda$ -sets with the orderings induced by reduced decompositions of  $\widehat{w}$ . Essentially, the description is given in terms of the “triangle triples”  $\tilde{\alpha}, \tilde{\alpha} + \tilde{\beta}, \tilde{\beta}$ . Say,  $\tilde{\alpha}, \tilde{\beta} \in \lambda(\widehat{w}) \Rightarrow \tilde{\alpha} + \tilde{\beta} \in \lambda(\widehat{w})$  and the latter root must be between  $\tilde{\alpha}$  and  $\tilde{\beta}$  if this set is treated as a sequence.

This theorem is not exactly new (although we cannot give precise references); however, we think, it is the most complete one of this kind. Several adjustments and generalizations were needed here since we need

to catch the ordering of the roots in  $\lambda(\widehat{w})$  and because we are doing the affine theory. It plays an important technical role in the paper; we decided to give its proof. It is mainly used as a list of properties of  $\lambda(\widehat{w})$ ; in the opposite direction, it gives that the intersections of  $\lambda$ -sequences with root subsystems remain  $\lambda$ -sequences.

*Main Theorem 2.4.* We need to know when a set of positive roots of a rank two subsystem inside a given *sequence*  $\lambda(\widehat{w})$  can be made consecutive using the Coxeter transforms in  $\lambda(\widehat{w})$ . This problem can be readily reduced to considering *triangle triples*  $\{\widetilde{\alpha}, \widetilde{\alpha} + \widetilde{\beta}, \widetilde{\beta}\}$  provided special conditions (a,b,c) from the theorem. Only for affine  $A_n, B_2, C_2, G_2$  or when  $|\widetilde{\alpha}| \neq |\widetilde{\beta}|$  the answer is always affirmative. For affine  $A_n$ , it can be deduced directly from the interpretation of [C2]) in terms of the lines on the cylinder. Generally the *admissibility* condition is needed, which is formulated in terms of subsystems of  $\widehat{R}$  of types  $B_3, C_3, D_4$ ; the justification requires using subsystems of types  $B_4, C_4, D_5$ . Some technical details of the proof of this theorem are omitted in the present paper. Namely, the obstacles of type  $D_4$  are not discussed in full and the proof is not complete in the case  $F_4$  (as  $|\widetilde{\alpha}| = |\widetilde{\beta}|$ ); we hope to continue this topic in other works.

*Its applications.* Presumably, this theorem “explains” the difficulties with “AHA-DAHA highest vectors”, generalizing Zelevinsky’s segments (known only in the  $A$ -case and for some root systems of small ranks). Generally, we need the approach from [KL1] and similar *geometric* constructions. The classification of *semisimple* representations of DAHA seems a natural first step toward the theory of highest vectors/weights.

In the case of affine root system of type  $A$ , the classification of such representations was obtained in [C9] in terms of infinite periodic skew Young diagrams (see also [C12]). Let us also mentioned paper [SuV], where the result is the same as in [C9] but the approach is somewhat different. In [C9], it was obtained as a corollary of the Main Theorem there (for arbitrary root systems) based on the technique of intertwiners combined with Zelevinsky’s classification in the case of the affine Hecke algebra of type  $A$  (adjusted to the semisimple affine case in author’s papers).

Theorem 2.4 for the affine  $A$ -system is an important (actually, the key) ingredient of the construction from [C9] and its justification. The classification of the semisimple representations of DAHA for arbitrary

root systems (not finished at the moment) will be addressed in author's future papers.

In this paper, we do not particularly need Theorem 2.4 for the theory of  $\mathcal{V}$ . Corollary 3.6 can be used to justify applying Key Lemma 12.5 to the irreducibility of  $\mathcal{V}$  (Main Theorem 12.7), but Lemma 1.2 is actually sufficient.

**0.2. Affine exponents.** The classical exponents of an irreducible reduced root system  $R$  are given by the formula:

$$\Pi_R(t) \stackrel{\text{def}}{=} \prod_{\alpha \in R_+} \frac{1 - t^{1+(\alpha, \rho^\vee)}}{1 - t^{(\alpha, \rho^\vee)}} = \prod_{i=1}^n \frac{1 - t^{m_i+1}}{1 - t}.$$

We will not review their various and important applications in algebra, combinatorics, geometry and topology; see, e.g., [B, Hu] concerning basic (certainly not all) aspects of their theory. Algebraically, this formula is about reducing coinciding terms in the numerator and denominator of the product in the left-hand side. This viewpoint will be the main in this paper.

The affine counterpart of  $\Pi_R$  is defined in terms of  $q$  and  $t_\alpha = q_\alpha^{k_\alpha}$ , where  $q_\alpha = q^{\nu_\alpha}$  for  $\nu_\alpha = (\alpha, \alpha)/2$  and  $k_\alpha$  depends only on  $\nu_\alpha$ :

$$\Pi_{\tilde{R}} \stackrel{\text{def}}{=} \prod_{\alpha \in R_+} \left( (1 - q_\alpha^{k_\alpha + (\alpha^\vee, \rho + \rho_k)}) \prod_{j=1}^{(\alpha^\vee, \rho)} \frac{(1 - q_\alpha^{j-1+k_\alpha + (\alpha^\vee, \rho_k)})}{(1 - q_\alpha^{j-1 + (\alpha^\vee, \rho_k)})} \right).$$

The normalization here is  $\nu_\alpha = 1$  for short roots;  $2\rho_k = \sum_{\alpha \in R_+} k_\alpha \alpha$ . Generally, the definition depends on a pair of affine extensions of  $R$  and/or its dual  $R^\vee$ . The definition above is for the pair  $\{\tilde{R}, \tilde{R}\}$  where  $\tilde{R} = \{[\alpha, \nu_\alpha j], \alpha \in R, j \in \mathbb{Z}\}$ .

The *root subsystem*  $\tilde{R}^0 = \{[\alpha, \nu_\alpha j], q_\alpha^{j + (\alpha^\vee, \rho_k)} = 1\}$  (compare with the denominator of the last formula), the Weyl group  $\widetilde{W}^0$  of  $\tilde{R}^0$ , and the relative Bruhat ordering for the pair  $\tilde{R}^0 \subset \tilde{R}$  play the key role in our analysis of  $\mathcal{V}$ . The main technical reason is that the roots from  $\tilde{R}^0$  lead to *singular intertwiners* (see below);  $\tilde{R}^0$  appears virtually in all statements and formulas of the paper.

We calculate  $\Pi_{\tilde{R}}$  in terms of the *affine exponents* in a way similar to the classical formula;  $\Pi_{\tilde{R}}$  becomes  $\Pi_R$  as  $q = 0$ ,  $t = t_{\text{sht}} = t_{\text{lng}}$ , (equivalently,  $k = k_{\text{sht}} = \nu_{\text{lng}} k_{\text{lng}}$ ), so our construction is a direct generalization of the classical definition.



In the simply-laced case:

$$\Pi_{\tilde{R}} = \prod_{i=1}^n \frac{\prod_{j=0}^{m_i} (1 - q^j t^{m_i+1})}{1 - t}.$$

We give complete lists of the affine exponents for any (reduced) root systems but their combinatorics is not considered in full in this paper, as well as their various applications. For instance, a Langlands-type duality holds, namely, Theorem 8.2:

$$\Pi_{\widehat{R}}(q, k_{\text{lng}}, k_{\text{sht}}) = \Pi_{\widehat{R}^\vee}(q, k_{\text{sht}}, k_{\text{lng}})$$

for the affine root system  $\widehat{R} = \{[\alpha, j], \alpha \in R, j \in \mathbb{Z}\}$  when the pair  $\{\widehat{R}, \widehat{R}^\vee\}$  is used in the definition of  $\Pi_{\widehat{R}}$ .

It is directly connected with the DAHA–Fourier transform, to be discussed in further papers. Generally, there are confirmations that, as far as DAHA can be used, Langlands’ correspondence can be associated with the Fourier transform, which plays the key role in the theory of DAHA.

The functoriality of  $\Pi_{\tilde{R}}$  and  $\Pi_{\widehat{R}}$  with respect to the affine root subsystems of  $R$  is not discussed at all. The *affine exponents* for  $\Pi_{\widehat{R}}$  satisfy a  $q \leftrightarrow q'$ –duality, for instance, rational singular  $k$  are in a sense dual to the zeros of the Poincaré polynomial (see (12.41)). This property is related to Theorem 12.7, but is not discussed systematically. Also, no interpretation of  $\Pi_{\tilde{R}}$  and  $\Pi_{\widehat{R}}$  as a Poincaré series is known.

The main motivation of the affine exponents in the paper is that their zeros upon multiplicative translations by the elements from  $q^{\mathbb{Z}}$  constitute the list of all *singular*  $t$ , defined as those making the radical of the polynomial representation nonzero. For instance,  $t_{\text{sing}} = q^{-j/(m_i+1)}$  for  $j > 0$  in the simply-laced case (the zeros of  $\Pi_R$  must be excluded). It is intimately related to the following formula.

**Calculating  $D_Y(D_X)(t^{-\rho})$ .** This expression generalizes the one in the rational setting, which is  $D_y(D_x)$  for the  $y$ –discriminant  $D_y = \prod_{\alpha \in R_+} y_\alpha$  in terms of the differential *rational Dunkl operators* [D],

$$y_b = \partial_b + \sum_{\alpha \in R_+} \frac{k_\alpha(b, \alpha)}{x_\alpha} (1 - s_\alpha) \quad \text{for } b \in P,$$

applied to the  $x$ –discriminant  $D_x = \prod_{\alpha \in R_+} x_\alpha$ .

The latter expression is already a constant (depending on the  $k$ -parameters), so the evaluation is not necessary in the rational case. This formula is due to Opdam [O1] in the crystallographical case; see [O2, DJO] for explicit formulas in the non-simply-laced cases and for  $I_2(2m)$ ,  $H_3$ ,  $H_4$ . A straightforward algebraic verification of this formula is known (and quite involved) only in the  $A$ -case (Dunkl, Hanlon).

The methods involved in [O1, O2, DJO] are the Heckman – Opdam theory of Jacobi–Jack polynomials, the Macdonald – Mehta conjecture (proved by Opdam) and also the semisimplicity theorem for the classical Hecke algebras from [GU]. Employing the Jack polynomials is remarkable; it requires differential-trigonometric setting (we use the difference-trigonometric setting in a similar manner). Using the monodromy method and [GU] provides a universal tool, but the least direct.

*The difference case.* The calculation of  $D_Y(D_X)$  becomes simple and uniform (for any  $k$ -parameters) in the general  $q, t$ -case. See Main Theorem 7.2. We check that  $D_Y(D_X)$  is proportional to the symmetric Macdonald polynomial of weight  $\rho$  using the *shift operator* from [C4] and employ the Macdonald’s principal value conjecture proven in [C9]; the result is  $\Pi_{\tilde{R}}$  up to some powers of  $q, t$ . Combining this formula with the limiting procedure from the general DAHA to its rational degeneration gives another justification of the Opdam formula (in the crystallographic case).

Concerning the evaluation (principal value) formula, its proof is entirely conceptual, a direct corollary of the DAHA–duality. We improve its deduction in this paper; see Proposition 6.6.

Note that the Macdonald polynomials collapse in the rational limit; the differential rational Dunkl operators are nilpotent in the polynomial representation and have no eigenvalues but 0. In the trigonometric case, the Jack polynomials exist but the duality collapse (the evaluation formula holds); We think, it explains why the difference theory is the most relevant to deal with  $D_Y(D_X)$  and, correspondingly,  $D_y(D_x)$ .

Last but not the least, the *Coxeter exponents* “naturally” enter the  $q, t$ -setting, almost directly due to their definition via  $\Pi_R$ . It “explains” why the Coxeter exponents appear almost everywhere in the Macdonald – Matsumoto theory of  $p$ -adic spherical functions; the limit  $q \rightarrow 0$  is exactly the passage from DAHA to the  $p$ -adic theory. Another advantage of the  $q, t$ -theory is that the affine exponents, in contrast to

their rational limits, are generally of multiplicity one. The multiplicities of the roots of the Bernstein-Sato polynomials are important in their general theory.

Generally, no simple ways can be expected for obtaining the Coxeter exponents from *rational*-differential operators; say, it was a difficult conjecture (proved by Opdam) that the Bernstein-Sato polynomials for the discriminant of  $R$  are given in terms of  $\{m_i\}$ . A metamathematical reason for it is trigonometric nature of their (main) definitions in terms of the  $W$ -invariant *Laurent* polynomials and via the product  $\Pi_R$ .

**The radical.** The zeros of the *rational*  $D_y(D_x)$  give the values of  $k$  when the discriminant  $D_x$  belongs to the *radical* of the evaluation pairing (rational evaluation, at 0). For such  $k$ , the radical,  $Rad_{rat}$ , is obviously nonzero; therefore the polynomial representation,  $\mathcal{V}_{rat}$ , is reducible. Vice versa, Opdam establishes that the irreducibility of  $\mathcal{V}_{rat}$  occurs exactly when the (rational) radical  $Rad_{rat}$  is nonzero, that happens at the zeros of  $D_y(D_x)$  up to their translations by negative integers.

The equivalence  $\{Rad_{rat} = \{0\}\} \Leftrightarrow \{\text{irreducibility of } \mathcal{V}_{rat}\}$  is very simple. Indeed, it is obvious in the  $\Leftarrow$  direction. If  $\mathcal{V}'_{rat}$  is a submodule of  $\mathcal{V}_{rat}$  then it contains a  $y$ -eigenvector  $v'$  (the eigenvalue can be only 0 for the rational DAHA) and therefore  $v' - v'(0)1$  belongs to  $Rad_{rat}$ .

*Main Theorem 12.7.* A modified variant of this reasoning can be used in the  $q, t$ -setting. Generally,  $\mathcal{V}$  can be reducible when  $Rad = \{0\}$  in the non-simply-laced case; it is of course not true any longer that all  $Y$ -eigenvalues in  $\mathcal{V}$  coincide. The list of exceptional cases, when  $Rad = \{0\}$  but  $\mathcal{V}$  is reducible, is given in Theorem 12.7. Its justification involves combinatorial case-by-case analysis.

*Etingof's theorem.* Recently Etingof [Et] obtained the first “half” of this list using the rational case and his general reduction theory. Actually he obtained it for the *degenerate* DAHA, the trigonometric limit of the general  $q, t$ -DAHA. The relation between our  $q, t$ -theorem and his one in the degenerate (trigonometric) case is as follows.

The parameter  $q$ , the center charge, is assumed generic in Theorem 12.7. However this assumption is not quite sufficient to connect the  $q, t$ -DAHA with its trigonometric degeneration. One has to impose furthermore that  $q^a t^b = 1 \Rightarrow a + kb = 0$  as  $t = q^k$  for  $a, b \in \mathbb{Q}$  in the simply-laced case (with two  $t$  here for  $B, C, F, G$ ). Then the reducibility

of  $\mathcal{V}$  and its (trigonometric) degeneration will occur exactly at the same  $k$ . This corresponds to the list from (12.37) in Theorem 12.7.

We mention that Etingof obtained (12.37) practically independently of this paper. Some exceptional cases when  $\mathcal{V}$  is reducible in spite of  $Rad = \{0\}$  were known to him, however he arrived at their *complete* list without knowing/using the methods and exact results of this paper.

Paper [Et] contains simple and *conceptual* interpretation of conditions from (12.37). We found this list on the basis of technical Lemma 12.1, which includes a case-by-case analysis. Actually, this lemma is directly connected with the *zigzag connectivity* from Lemma 12.3 that seems quite general. However, the relation of (12.37) to the Borel – de Siebenthal algorithm found by Etingof is remarkable and clarifying. It is good to have alternative approaches to such problem that seem of fundamental nature.

We would like to mention the paper [L] in this context; considering the “mixed” products of the normalized intertwiners and the  $T$ -elements ( $T_w^c$  from Proposition 8.1 there) is similar to our approach. However, the main problem in the theory of polynomial representation is dealing with *non-invertible intertwiners*, where the counterparts of Lusztig’s  $T_w^c$  may depend on the particular choice of the reduced decomposition.

The second “half” of the exceptional cases, namely (12.39), describes the reducibility of  $\mathcal{V}$  with zero radical without imposing the assumption  $q^a t^b = 1$ ,  $a, b \in \mathbb{Q} \Rightarrow a + kb = 0$ . It is, in a sense, *dual* to (12.37), so it is likely that the whole Theorem 12.7 can be “deduced” from the rational case using Etingof’s approach at greater potential.

Our approach is expected to be applicable to describing the cases when  $\mathcal{V}/Rad$  is not irreducible (it naturally includes those from Theorem 12.7), but we did not establish Lemma 12.3 and related stuff in proper generality so far. It was proven in [C11] that  $\mathcal{V}/Rad$  is always irreducible for generic  $q$  if this quotient is *finite-dimensional*. The technique of intertwiners (in the  $q, t$ -setting) can be used to address this problem in general.

**0.3. Non-semisimple polynomials.** A natural approach here is to use the decomposition of the polynomial representation  $\mathcal{V}$  for generic  $q, t$  in terms of the non-symmetric Macdonald polynomials  $E_b$ ,  $b \in P$ ,

and tend  $t$  (or  $k$ ) to a *singular* values  $t_{\text{sing}} = q^{k_{\text{sing}}}$ . Such decomposition of  $\mathcal{V}$  and the related technique of intertwiners are well known for generic  $q, t$ ; see [C7] ([KS] in the  $A$ -case) and [M6, C12]. However there are no reasonable formulas for (arbitrary) coefficients of  $E_b$  and no straight ways to control directly the limits of these polynomials coefficient-wise as  $t$  becomes singular (unless for  $A_1$  and in some rank 2 examples).

A bypass is in representing  $\mathcal{V}$  as a sum of a system of finite dimensional subspaces  $\mathcal{V}_b$  such that their limits  $t \rightarrow t_{\text{sing}}$  can be calculated *exactly* and the corresponding “Gr” is a direct sum of one-dimensional subspaces, i.e, for  $\{\mathcal{V}_b, b \in P\}$  constituting a *maximal system* of vector spaces generating  $\mathcal{V}$ .

Here the limit is understood in the sense of vector bundles over a curve (dimension one is important). *All* regular (at  $t_{\text{sing}}$ ) linear combinations of vectors with the coefficients in the field of rational functions in terms of  $(t - t_{\text{sing}})$  must be considered. Such limiting procedure preserves the dimensions and, given  $b$ , extends the vector bundle  $\{\mathcal{V}_b(t)\}$  from generic  $t$  to singular  $t_{\text{sing}}$ . Generally, there can be many choices of *one-parametric* limiting procedures, but this construction is independent of such choices.

*Main Theorem 9.1.* It contains a construction of such system of spaces. Actually, we simply give a uniform definition of  $\mathcal{V}_b$  for all  $t$  (including  $t_{\text{sing}}$ ) and check that the dependence of  $t$  is flat. The description of spaces  $\mathcal{V}_b$  is very explicit; they are given in terms of  $\tilde{R}^0$ , the subsystem of singular roots for a given  $t_{\text{sing}}$ . This construction is entirely combinatorial in terms of  $\tilde{R}^0$  ( $t$  is arbitrary, possibly different from  $t_{\text{sing}}$ ).

The first application is that the Macdonald polynomial  $E_b$  for  $b \in P$  exists at  $t_{\text{sing}}$  if (*and only if* in some sense)  $\dim \mathcal{V}_b = 1$ , which is a pure combinatorial condition concerning  $b$  and  $\tilde{R}^0$ .

Given  $b \in P$ , the definition of the space  $\mathcal{V}_b$  requires a reduced decomposition of  $\pi_b \in \widehat{W}$ , the *reduction of  $b$  modulo  $W$* . To be exact,  $\pi_b$  is defined as the element of minimal length in the coset  $bW \in \widehat{W}$  (it is unique). The space  $\mathcal{V}_b$  does not depend on the particular choice of the reduced decomposition of  $\pi_b$  and can be calculated in terms of the *relative Bruhat ordering* defined for  $\tilde{R} \bmod \tilde{R}^0$ . This space consists of *generalized  $Y$ -eigenvectors* for the eigenvalue  $-b_{\sharp}$  (see the definition in Proposition 6.2); generally, not all of them. Its dimension is always finite even if  $q$  is a root of unity.

We note that the elements  $\{\pi_b\}$  and the standard Bruhat ordering on  $\widehat{W}$  govern the combinatorics of *affine Schubert manifolds*, although  $\widetilde{R}^0$  does not appear in the theory of the affine Grassmanian and such manifolds.

*Non-semisimple Macdonald polynomials.* They are top polynomial  $\widetilde{E}_b \in \mathcal{V}_b$ . The construction is explicit;  $\widetilde{E}_b$  is given in terms of the *generalized chain of intertwiners* (discussed below) corresponding to a given reduced decomposition of  $\pi_b$ .

The polynomials  $\widetilde{E}_b$  are not unique and may depend on the choice of this decomposition modulo polynomials from “lower”  $\mathcal{V}_c$ , although the actual flexibility of this definition is limited. In many cases they are determined uniquely even if  $\dim \mathcal{V}_b > 1$ . These polynomials have correct leading terms and form a basis of  $\mathcal{V}$ ;  $\widetilde{E}_b = E_b$  if  $\dim \mathcal{V}_b = 1$ .

*Generalized chains.* The  $Y$ -intertwiners corresponding to simple  $\alpha_i$  ( $i > 0$ ) are  $(T_i + c(Y_{\alpha_i}))$  for nonaffine simple roots  $\alpha_i \in R_+$  (treated as vectors in  $P$ ) and  $c(x) = (t_i^{1/2} - t_i^{-1/2})/(x^{-1} - 1)$ . It is more involved for the affine  $\alpha_0$ ; namely, following [C7], we need to consider  $\tau_+(T_0)$  instead of  $T_0$  for the automorphism  $\tau_+$  of DAHA (formally) corresponding to multiplication by the Gaussian. Here  $c(Y_{\alpha_i})$  may become infinite in a chain of intertwiners; we call the corresponding places *singular* in the paper, as well as corresponding simple reflections and roots from  $\lambda(\pi_b)$ .

We define generalized chains of intertwiners by replacing singular simple intertwiners  $(T_i + \infty)$  by  $T_i$ . See (9.5), (9.6) for exact definitions. Choosing  $T_i$  here ensures the proper leading terms of the non-semisimple Macdonald polynomials. Another motivation is that the relations between the intertwiners and  $T$  are of fundamental importance (note a connection with virtual links).

Two natural problems arise:

- (a) determining how the non-semisimple polynomials  $\widetilde{E}_b$  depend on the choice of the reduced decompositions of  $\pi_b$ , which is mainly covered by Theorem 5.2;
- (b) finding “large” families of  $b \in P$  such that  $\dim \mathcal{V}_b = 1$  and therefore  $\widetilde{E}_b = E_b$ , which is addressed in Proposition 10.1 and in the following theorem.

*Main Theorem 10.4.* It contains efficient tools for solving (b) and exact calculating the spaces  $\mathcal{V}_b$  and the spaces  $\mathcal{V}(-b_{\sharp})^{\infty}$ . The latter spaces are defined as the spaces of *all generalized eigenvectors* with the

$Y$ -eigenvalue  $-b_{\sharp}$  serving  $\tilde{E}_b$ ; see the definition in Proposition 6.2. We analyze what happens with  $\mathcal{V}_c$  if  $\pi_c$  is replaced by  $\pi_c \hat{w}$  for arbitrary  $\hat{w} \in \widehat{W}$ , then do it for  $\hat{w} \in \widehat{W}$  from the centralizer of the weight  $-0_{\sharp} = \rho_k$  (the  $Y$ -eigenvalue of  $1 \in \mathcal{V}$ ) and, finally, for  $\hat{w}$  from  $\widetilde{W}^0$ , the Weyl group of  $\widetilde{R}^0$ .

An application of Proposition 10.1 and Theorem 10.4 is Theorem 10.3 about the canonical semisimple submodule  $\mathcal{V}_{ss} \subset \mathcal{V}$ . It generalizes the construction from [Ka] in the  $A$ -case based, in its turn, on paper [FJMM] where the symmetric case was considered.

We consider the  $A$  case in detail because it is important to establish the connection with these papers. However we do not give the final list of root systems and parameters  $k$  when  $\mathcal{V}_{ss}$  is nonzero. Obtaining these conditions and an explicit description of  $\mathcal{V}_{ss}$  does not look very difficult, but involving other methods is more reasonable here.

To be more precise, we do not touch in the paper the *wheel condition* from [FJMM, Ka], an alternative way (better to say, complementary) to introduce  $\mathcal{V}_{ss}$  as an ideal in the polynomial representation. Counterparts of the wheel condition can be obtained for (many, maybe all) root systems within the technique of this paper; actually, only the evaluation pairing is necessary. However we prefer to postpone with finalizing the consideration of  $\mathcal{V}_{ss}$  until future papers on the Jantzen-type filtration. We only mention here that the Kasatani conjecture [Ka], verified in [En] via the localization functor, can be almost certainly managed using *directly* the technique of intertwiners.

Generally, it is expected that the Jantzen-type filtration gives a natural way to decompose  $\mathcal{V}$  with the constituents that are irreducible in many cases, including the  $A$ -case. We consider only two examples in this paper, the “highest” and the “lowest” constituents, namely, the quotient  $\mathcal{V}/Rad$  and the submodule  $\mathcal{V}_{ss}$ . Mainly we discuss  $\mathcal{V}$  subject to  $Rad = \{0\}$ , but the same technique can be used for  $\mathcal{V}/Rad$ .

**Further topics.** The spaces  $\mathcal{V}_b$  are not the smallest “natural” space of generalized eigenvectors corresponding to the eigenvalue  $-b_{\sharp}$  that contain  $\tilde{E}_b$ . Following the chain of intertwiners for  $\tilde{E}_b$ , we define the spaces  $\tilde{E}_b \in \tilde{V}_b \subset \mathcal{V}_b$ , that form a *maximal system* of subspaces in  $\mathcal{V}$ , like  $\{\mathcal{V}_b\}$ , with  $\{\tilde{E}_b\}$  as top polynomials. The definition is more direct than that for  $\{\mathcal{V}_b\}$ : given  $b$ , the space  $\tilde{V}_b$  is given *exactly* in terms of the reduced decomposition of  $\pi_b$  that is used for  $\tilde{E}_b$ . However, in contrast



to  $\mathcal{V}_b$ , this space may depend on a particular choice of the reduced decomposition.

The spaces  $\mathcal{V}_b$  and  $\tilde{V}_b$  are  $Y$ -modules. The space  $\tilde{V}_b$  is  $Y$ -cyclic in many cases; the strongest result in this direction we have is Proposition 10.12 based on the following theorem.

*Main Theorem 10.6.* We prove that  $\tilde{V}_b$  considered as a  $Y$ -module is *covered* by a certain  $Y$ -module defined explicitly in terms of the generators and relations with the structural constants that are essentially *integers*. It is defined via the Demazure operators and is connected with Schubert polynomials. Due to such integrality, we can switch here from DAHA to any degeneration we wish. The proof of the theorem is a straightforward calculation; it gives a natural direct approach to the  $Y$ -cyclicity of  $\tilde{V}_b$ .

In a sense,  $\tilde{V}_b$  is cyclic “almost always”. To be more exact, if the set  $\lambda(\pi_b)$  is sufficiently small, then  $\tilde{R}^0$  contains only pairwise orthogonal roots and we can use Corollary 10.7. If  $\lambda(\pi_b)$  is sufficiently large, then Main Theorem 2.4 can be applied to collect all singular roots in  $\lambda(\pi_b)$  in a connected segment using the Coxeter transforms and Proposition 10.12 can be employed.

The  $Y$ -cyclicity of  $\tilde{V}_b$  (if known) can certainly simplify using the technique of intertwiners for  $\mathcal{V}$  and other induced modules where it holds. Concerning general  $Y$ -induced modules (free modules induced  $Y$ -eigenvectors), we need them only a little in this part of the paper. Generalizing  $\tilde{V}_b$ , we define the spaces  $\tilde{V}_{\hat{w}}$ ; note that using  $\hat{w} = \pi_b$  only is a special feature of  $\mathcal{V}$ . The modules  $\tilde{V}_{\hat{w}}$  are *cyclic* for sufficiently large  $\lambda(\hat{w})$  (not one-dimensional as for  $\mathcal{V}$ ); see Proposition 10.12.

*Irreducibility of  $\mathcal{V}$ .* The end of the paper is devoted to Main Theorem 11.8 on irreducibility of  $\mathcal{V}$  subject to the condition  $Rad = \{0\}$ , which demonstrates almost all aspects of the technique of intertwiners.

The condition  $Rad = \{0\}$  readily gives that all three spaces  $\tilde{V}_b \subset \mathcal{V}_b \subset \mathcal{V}(-b_{\sharp})^{\infty}$  contain a *unique* Macdonald polynomial  $E_{b^{\circ}}$ , where  $b^{\circ} \in P$  is defined combinatorially in terms of  $b$ . It suffices to consider the chains of intertwiners that end at  $E_b$  (i.e., for  $b = b^{\circ}$ ). Only non-invertible intertwiners may lead to reducibility. Moreover,  $Rad = \{0\}$  implies that intertwiners of type  $(T - t^{1/2})$  cannot appear when the chain goes from  $E_b$  to  $E_c$  avoiding *singular* intertwiners. Assuming that  $\mathcal{V}' \subset \mathcal{V}$  is a proper DAHA submodule, we proceed as follows.



Given a chain of intertwiners, let  $E_b$  be the first Macdonald polynomial in  $\mathcal{V}'$ . Then the previous one is  $E_a \notin \mathcal{V}'$  and the intertwiner between them can be only of type  $(T + t^{-1/2})$ .

We go from  $E_b$  until the first *singular* intertwiner *if it exists* (not always). Let  $E_c$  be the last in this chain before this place for a reduced decomposition extending that for  $\pi_a$ :  $\lambda(\pi_a) \subset \lambda(\pi_c)$ . Then we apply the corresponding  $T$  to  $E_c$  and then go back following the sequence  $\lambda(\pi_c) \setminus \lambda(\pi_a)$  taken in the opposite order, from the last root to the first.

The resulting polynomial (the “end” of this chain) will belong to  $\mathcal{V}' \cap \mathcal{V}(-a_\#)^\infty$ . The analysis shows that  $Rad = \{0\}$  implies that it must be proportional to  $E_a$  with a nonzero coefficient of proportionality. Therefore  $E_a$  belongs to  $\mathcal{V}'$ , a contradiction.

This method, *reflection at the first singular place*, is actually of general nature and can be applied in various situations. For the first time in the DAHA context it was used in [CO] for  $A_1$ , where it gives a complete decomposition of  $\mathcal{V}$  including the cases of roots of unity. Let us discuss some combinatorial aspects of this method.

*Zigzags.* In this proof, Lemma 11.1 is needed for managing the situation when there are several intertwiners of type  $(T + t^{-1/2})$  in  $\lambda(\pi_c) \setminus \lambda(\pi_b)$ . For coinciding  $t$  (as  $k_{\text{sht}} = \nu_{\text{lng}} k_{\text{lng}}$ ), this lemma is essentially on the combinatorics of the sets  $R_{ht}$  of roots  $\alpha \in R_+$  with fixed  $ht \stackrel{\text{def}}{=} (\alpha, \rho^\vee)$ . We introduce the *zigzags* alternating between  $R_{ht}$  and  $R_{ht+1}$  with the *links* (as  $t_{\text{lng}} = t_{\text{sht}}$ ) corresponding to adding or subtracting simple roots, quite a classical matter. The claim is that *any maximal zigzag contains at least one endpoint from  $R_{ht}$* .

This claim is actually a combinatorial variant of the formula for  $\Pi_R$ . If the  $k$ -parameters are arbitrary, then smaller subsets must be considered instead of  $R_{ht}$  and *links* become somewhat more involved; however the claim about the endpoints holds. This approach is expected to give a description of the cases when  $\mathcal{V}/Rad$  is not irreducible.

Generally, the technique of intertwiners alone does not seem sufficient for decomposing  $\mathcal{V}$  and becomes too combinatorial even for managing the irreducibility of  $\mathcal{V}$  subject to  $Rad = \{0\}$ . We decided to omit the details of the zigzag construction in this paper.

**0.4. Expected applications.** Decomposing the polynomial representation  $\mathcal{V}$  is of key importance in the theory of DAHA and is expected to have many applications. In the rational case, the theory of  $\mathcal{V}$  was

started by Opdam; see [DO, DJO]. From the viewpoint of applications, the general  $q, t$ - case seems the most fruitful, although the degenerate cases have important applications too.

As a DAHA-module,  $\mathcal{V}$  is a *universal spherical representation*; this alone is sufficient to study it thoroughly. Moreover, its identification with the algebra of  $X$ -polynomials makes  $\mathcal{V}$  and all its constituents *commutative algebras*. The quotients possess the unit, other constituents do not. There is also a natural projective action of  $PSL_2(\mathbb{Z})$  on *finite dimensional* constituents of  $\mathcal{V}$  (maybe reducible) subject to certain technical restrictions, simple to control.

The latter action exists because of the following conceptual reason. The automorphism  $\tau_-$ , one of two generators  $\tau_{\pm}$  of the projective  $PSL_2(\mathbb{Z})$ , is an outer automorphism of DAHA formally corresponding to multiplication (conjugation) by the  $Y$ -Gaussian. It *always* acts in  $\mathcal{V}$  and its constituents; see Proposition 6.3. However, the generator  $\tau_+$ , represented by the  $X$ -Gaussian (an infinite Laurent series) may act only in finite dimensional constituents of  $\mathcal{V}$  (the condition  $\dim < \infty$  is not always sufficient).

The expectations are that  $\mathcal{V}$  and its quotients serve quite a few examples of monoidal categories (with tensoring), especially if the action of  $PSL_2(\mathbb{Z})$  is present there, and go well beyond such examples.

The celebrated Verlinde algebras are the key example. They are interpreted in the DAHA theory as symmetric subalgebras of *perfect* quotients of  $\mathcal{V}$  as  $q$  is a root of unity in the so-called group case  $t = q$  (that is the simplest possible setting in the DAHA theory apart from  $t = 1$ ).

The multiplication in  $\mathcal{V}$  leads to the *fusion* of integrable Kac–Moody modules. The DAHA-action of  $PSL_2(\mathbb{Z})$  is nothing but the action of Verlinde operators  $S, T$ ;  $T$  is multiplication by the  $X$ -Gaussian,  $S$  becomes the DAHA–Fourier transform.

Using the terminology from [C12], symmetrizations of *perfect representations* are natural generalizations of the Verlinde algebras. However, the perfect representations are well beyond the usual Verlinde algebras. Let us comment on it.

First, one can take any  $t$  instead of  $t = q$  provided that the corresponding perfect representation exists; this means that the characters in the Verlinde theory will be replaced by the symmetric Macdonald polynomials (treated as functions on at certain finite sets of points).

Second, the *non-symmetric* Macdonald polynomials can be considered here instead of their symmetrization. Furthermore, the *non-semisimple* Macdonald polynomials can be taken, corresponding to *non-semisimple* counterparts of perfect representations.

Third, it is not necessary to assume that  $q$  is a root of unity (one of the main discoveries of the DAHA approach); an important part of the theory of perfect representations is for generic  $q$  (then  $k$  must be *singular*). They are finite dimensional and has all key structures of Verlinde algebras.

**Possible relations.** The following is a sketch of (some) known and expected applications of  $\mathcal{V}$  and its constituents.

(a) Presumably all *Verlinde-type algebras*, describing “fusion” of integrable modules for the Kac-Moody algebras, Virasoro algebras,  $\mathcal{W}$ -algebras and similar objects, are quotients or constituents of  $\mathcal{V}$  (see confirmations in [FHST],[MTi]). Moreover, infinite dimensional constituents of  $\mathcal{V}$  are expected to be connected with the theory at arbitrary Kac-Moody central charge  $c$ . The most interesting case  $|q| = 1$  when  $q$  is *not* a root of unity (then  $c \in \mathbb{R}$ ) may lead to the *Kac-Moody  $L^2$ -theory*, generalizing the classical harmonic analysis on symmetric spaces. Straight attempts to create such theory were unsuccessful.

(b) Similar expectations are for the tensor category of *all* representations of Lusztig’s quantum group at roots of unity. The corresponding Verlinde algebra describes the *reduced subcategory* of this category; the equivalence with the definition of Verlinde algebras in the Kac-Moody theory is due to Kazhdan, Lusztig [KL2] and Finkelberg. The first known examples of the *non-semisimple* Verlinde algebras look very similar to what can be expected in the so-called “case of parallelogram”. Note that the monoidal structure (fusion, tensoring) and the action of  $PSL_2(\mathbb{Z})$  are generally difficult problems for the complete Lusztig’s category.

(c) If an arbitrary *perfect representation* is taken, then no categorical interpretation is known and expected (there are no reasons for integrality and positivity of the structural constants). However, all such representations give important examples of Fourier transform theories satisfying *all* standard classical properties. This line *directly* generalizes Fourier transforms associated with the irreducible modules in the theory of Weyl algebras (non-commutative tori) at roots of unity. Among other applications, perfect representations are related to the

generalized Macdonald eta-type identities, Gaussian sums [C8] and the so-called diagonal coinvariants [Ha, Go, C10].

(d) Semisimple *submodules* of  $\mathcal{V}$  generalize the construction from [FJMM] of *ideals* in the ring of symmetric polynomials of type  $A_n$  linearly generated by symmetric Macdonald polynomials. These ideals are expected to be meaningful in the theory of  $\widehat{\mathfrak{gl}}_N$  and the corresponding  $\mathcal{W}$ -algebras, presumably, via the duality from [VV1] and [STU]. They also give some kind of restriction maps  $\mathfrak{gl}_M \subset \mathfrak{gl}_N$ , although DAHA generally do not have *straight* embeddings of this type (unless the approach from [BE] or similar methods involving completions of DAHA are used).

(e) The theory of  $\mathcal{V}$  is connected with the Plancherel formula on the *affine* Hecke algebra due to Macdonald (the spherical case), Matsumoto, Lusztig and many others, describing the decomposition of the regular representation of the affine Hecke algebra. The regular representation of AHA is interpreted as an induced module (depending on a generic weight) in the DAHA theory. Its spherical part is associated with the weight  $-\rho_k$  and can be identified with  $\mathcal{V}$ . The relation to [KL1, HO] is via a new theory of Jantzen-type filtration of DAHA considered in the limit  $q \rightarrow 0$ ; it may include applications to square integrable and tempered irreducible AHA-representations.

(f) In the rational case, *singular*  $k$  (when the radical becomes nonzero) correspond to singular multi-dimensional Bessel functions. The most degenerate case is when  $\mathcal{V}_{rat}$  has a finite dimensional quotient. Such quotients found important combinatorial applications, for example, in [Go]. They may be related to the minimal conformal theories based on Virasoro-type algebras and their “perturbations” (adding additional parameters). The  $q, t$ -case is connected with the rational theory in many ways; for instance, certain Verlinde algebras can be  $q$ -deformed and then identified with their rational limits [C8]. It may reflect various relations between the Kac-Moody and Virasoro theories.

(g) The exponential map from [C10] can be used, in principle, to establish a correspondence between the decomposition theory of the polynomial representation for the rational DAHA [DJO, DO] and that in the  $q, t$ -case, although this approach (generally) requires analytic setting. Recent [Et] is a step in this direction. Algebraically, the exponential map identifies finite dimensional DAHA modules and their rational degeneration. This map is connected with the localization functor, the

monodromy of a KZ-type connection, from [GGOR, VV2]. Using the exponential map and the KZ-monodromy for the polynomial and other infinite dimensional representations triggers interesting problems and certainly must be studied thoroughly.

(h) The evaluation pairing of  $\mathcal{V}$  plays the key role in the DAHA theory. For the Macdonald polynomials, it is given in terms of their values at certain “shifted lattices” of points (weights). These lattices of points are *exactly* those used in the theory of *interpolation polynomials* studied by Knop – Sahi, Okounkov – Olshantsky and others. The definition requires the “semisimplicity” of these sets (directly related to the semisimplicity of  $\mathcal{V}$ ). The consideration of the non-semisimple quotients of the polynomial representation in this paper may lead to the theory of “non-semisimple” interpolation polynomials. There are classical constructions that confirm that such theory must exist.

(i) In the semisimple case, the polynomial representation is completely governed by the intertwining operators; the latter give a representation of the affine Weyl group  $\mathcal{V}$ . The non-semisimple setting requires considering intertwiners *together* with the  $T$ -generators. The corresponding formalism is connected with the theory of virtual links in topology (when extended from  $A$  to arbitrary root systems). We note that only the DAHA-modules that do not involve non-invertible intertwiners (singular ones may appear) are *directly* connected with Kauffman’s axioms. In a somewhat different (but related) direction, the weight decompositions of semisimple DAHA modules may give something for the theory of “categorization”.

(j) The *localization functor*, the monodromy map of a KZ-type connection, from [GGOR] (the rational case) and [VV2] (the differential-trigonometric case) is an important motivation for the direction of this paper. For instance, the *affine exponents* are directly connected with the non-semisimplicity of non-affine Hecke algebras at roots of unity. Practically all problems discussed in the paper can be translated to the corresponding problems for Hecke algebras at roots of unity and/or for the Schur algebras. Generally, the geometric theory of Hecke algebras is better developed than that for DAHA. However, in the DAHA theory there is a greater potential of using relatively elementary (non-geometric tools) like those developed in this paper.

(k) Paper [BE] on the parabolic induction in the DAHA theory opens a systematic way for analyzing induced representations (including  $\mathcal{V}$ ).

It does not cover the  $q, t$ -setting, but, presumably, can be generalized. The representations parabolically induced from finite dimensional ones are in the focus of [BE]. Paper [VV3] gives a classification of finite dimensional modules in the spherical case for generic  $q$ ; it includes *perfect representations* and gives some non-semisimple ones. Before this paper, a complete classification of finite dimensional modules was obtained only for  $A$  in [BEG] (all appeared perfect). Understanding parallelism with the  $p$ -adic theory is a natural challenge.

**Acknowledgements.** The author is thankful to A. Garsia, D. Kazhdan, E. Opdam and N. Wallach for stimulating discussions. Special thanks to P. Etingof. The author is grateful to RIMS, Kyoto University, for the invitations in 2005, when the paper was started and the author delivered a series of lectures on the classification of semisimple DAHA modules of type  $A$ , and in 2007, when this paper was essentially completed. The author thanks M. Kashiwara and T. Miwa for their hospitality and also thanks IHES for the invitations in 2006-07.

## 1. AFFINE WEYL GROUPS

Let  $R = \{\alpha\} \subset \mathbb{R}^n$  be a root system of type  $A, B, \dots, F, G$  with respect to a euclidean form  $(z, z')$  on  $\mathbb{R}^n \ni z, z'$ ,  $W$  the **Weyl group** generated by the reflections  $s_\alpha$ ,  $R_+$  the set of positive roots ( $R_- = -R_+$ ) corresponding to fixed simple roots  $\alpha_1, \dots, \alpha_n$ ,  $\Gamma$  the Dynkin diagram with  $\{\alpha_i, 1 \leq i \leq n\}$  as the vertices.

We will also use the dual roots (coroots) and the dual root system:

$$R^\vee = \{\alpha^\vee = 2\alpha/(\alpha, \alpha)\}.$$

The root lattice and the weight lattice are:

$$Q = \oplus_{i=1}^n \mathbb{Z}\alpha_i \subset P = \oplus_{i=1}^n \mathbb{Z}\omega_i,$$

where  $\{\omega_i\}$  are fundamental weights:  $(\omega_i, \alpha_j^\vee) = \delta_{ij}$  for the simple coroots  $\alpha_i^\vee$ .

Replacing  $\mathbb{Z}$  by  $\mathbb{Z}_\pm = \{m \in \mathbb{Z}, \pm m \geq 0\}$  we obtain  $Q_\pm, P_\pm$ . Note that  $Q \cap P_+ \subset Q_+$ . Moreover, each  $\omega_j$  has all nonzero coefficients (sometimes rational) when expressed in terms of  $\{\alpha_i\}$ . Here and further see [B].

The form will be normalized by the condition  $(\alpha, \alpha) = 2$  for the *short* roots. This normalization coincides with that from the tables in [B] for the systems  $A, C, D, E, G$ . Thus,

$$\nu_\alpha \stackrel{\text{def}}{=} (\alpha, \alpha)/2 \text{ can be either } 1, \text{ or } \{1, 2\}, \text{ or } \{1, 3\}.$$

We will use the notation  $\nu_{\text{lng}}$  for the long roots ( $\nu_{\text{sht}} = 1$ ).

This normalization leads to the inclusions  $Q \subset Q^\vee, P \subset P^\vee$ , where  $P^\vee$  is defined to be generated by the fundamental coweights  $\omega_i^\vee$ .

Let  $\vartheta \in R^\vee$  be the *maximal positive coroot*. All simple coroots appear in its decomposition in  $R^\vee$ . It also belongs to  $R$ , i.e. is a root, because of the choice of normalization; so all simple roots appear in its decomposition in  $R$ .

Also note that  $2 \geq (\vartheta, \alpha^\vee) \geq 0$  for  $\alpha > 0$ ,  $(\vartheta, \alpha^\vee) = 2$  only for  $\alpha = \vartheta$ , and  $s_\vartheta(\alpha) < 0$  if  $(\vartheta, \alpha) > 0$ .

See [B] to check that  $\vartheta$  considered as a root is maximal among all short positive roots of  $R$ . It is also the least nonzero element in  $Q_+^+ = Q_+ \cap P_+ = Q \cap P_+$  with respect to  $Q_+$ .

Setting  $\nu_i = \nu_{\alpha_i}$ ,  $\nu_R = \{\nu_\alpha, \alpha \in R\}$ , one has

$$(1.1) \quad \rho_\nu \stackrel{\text{def}}{=} (1/2) \sum_{\nu_\alpha = \nu} \alpha = \sum_{\nu_i = \nu} \omega_i, \text{ where } \alpha \in R_+, \nu \in \nu_R.$$

Note that  $(\rho_\nu, \alpha_i^\vee) = 1$  as  $\nu_i = \nu$ . We will call  $\rho_\nu$  **partial**  $\rho$ .

**1.1. Affine roots.** The vectors  $\tilde{\alpha} = [\alpha, \nu_\alpha j] \in \mathbb{R}^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$  for  $\alpha \in R, j \in \mathbb{Z}$  form the **affine root system**  $\tilde{R} \supset R$  ( $z \in \mathbb{R}^n$  are identified with  $[z, 0]$ ). We add  $\alpha_0 \stackrel{\text{def}}{=} [-\vartheta, 1]$  to the simple roots for the **maximal short root**  $\vartheta$ . The corresponding set  $\tilde{R}_+$  of positive roots coincides with  $R_+ \cup \{[\alpha, \nu_\alpha j], \alpha \in R, j > 0\}$ .

We will sometimes write  $\tilde{R} = \tilde{A}_n, \tilde{B}_n, \dots, \tilde{G}_2$  when dealing with concrete affine root systems defined as above.

Any positive affine root  $[\alpha, \nu_\alpha j]$  is a linear combinations with non-negative integral coefficients of  $\{\alpha_i, 0 \leq i \leq n\}$ . Indeed, it is well known that  $[\alpha^\vee, j]$  is such combination in terms of  $\{\alpha_i^\vee, 1 \leq i \leq n\}$  and  $[-\vartheta, 1]$  for the system of affine *coroots*, that is  $\tilde{R}^\vee = \{[\alpha^\vee, j], \alpha \in R, j \in \mathbb{Z}\}$ . Hence,  $[-\alpha, \nu_\alpha j] = \nu_\alpha [-\alpha^\vee, j]$  has the required representation.

Note that the sum of the long roots is always long, the sum of two short roots can be a long root only if they are orthogonal to each other. This property gives another justification of the claim that  $\tilde{R}$  is a root system.

We complete the Dynkin diagram  $\Gamma$  of  $R$  by  $\alpha_0$  (by  $-\vartheta$ , to be more exact); it is called **affine Dynkin diagram**  $\tilde{\Gamma}$ . One can obtain it from the completed Dynkin diagram from [B] for the *dual system*  $R^\vee$  by



reversing all arrows. The number of laces between  $\alpha_i$  and  $\alpha_j$  in  $\tilde{\Gamma}$  will be denoted by  $m_{ij}$ .

The set of the indices of the images of  $\alpha_0$  by all the automorphisms of  $\tilde{\Gamma}$  will be denoted by  $O$  ( $O = \{0\}$  for  $E_8, F_4, G_2$ ). Let  $O' = \{r \in O, r \neq 0\}$ . The elements  $\omega_r$  for  $r \in O'$  are the so-called minuscule weights:  $(\omega_r, \alpha^\vee) \leq 1$  for  $\alpha \in R_+$ .

Given  $\tilde{\alpha} = [\alpha, \nu_\alpha j] \in \tilde{R}$ ,  $b \in P$ , let

$$(1.2) \quad s_{\tilde{\alpha}}(\tilde{z}) = \tilde{z} - (z, \alpha^\vee)\tilde{\alpha}, \quad b'(\tilde{z}) = [z, \zeta - (z, b)]$$

for  $\tilde{z} = [z, \zeta] \in \mathbb{R}^{n+1}$ .

The **affine Weyl group**  $\tilde{W}$  is generated by all  $s_{\tilde{\alpha}}$  (we write  $\tilde{W} = \langle s_{\tilde{\alpha}}, \tilde{\alpha} \in \tilde{R}_+ \rangle$ ). One can take the simple reflections  $s_i = s_{\alpha_i}$  ( $0 \leq i \leq n$ ) as its generators and introduce the corresponding notion of the length. This group is the semidirect product  $W \ltimes Q'$  of its subgroups  $W = \langle s_\alpha, \alpha \in R_+ \rangle$  and  $Q' = \{a', a \in Q\}$ , where

$$(1.3) \quad \alpha' = s_\alpha s_{[\alpha, \nu_\alpha]} = s_{[-\alpha, \nu_\alpha]} s_\alpha \quad \text{for } \alpha \in R.$$

The **extended Weyl group**  $\widehat{W}$  generated by  $W$  and  $P'$  (instead of  $Q'$ ) is isomorphic to  $W \ltimes P'$ :

$$(1.4) \quad (wb')([z, \zeta]) = [w(z), \zeta - (z, b)] \quad \text{for } w \in W, b \in P.$$

From now on,  $b$  and  $b'$ ,  $P$  and  $P'$  will be identified.

Given  $b \in P_+$ , let  $w_0^b$  be the longest element in the subgroup  $W_0^b \subset W$  of the elements preserving  $b$ . This subgroup is generated by simple reflections. We set

$$(1.5) \quad u_b = w_0 w_0^b \in W, \quad \pi_b = b(u_b)^{-1} \in \widehat{W}, \quad u_i = u_{\omega_i}, \quad \pi_i = \pi_{\omega_i},$$

where  $w_0$  is the longest element in  $W$ ,  $1 \leq i \leq n$ .

The elements  $\pi_r \stackrel{\text{def}}{=} \pi_{\omega_r}$ ,  $r \in O'$  and  $\pi_0 = \text{id}$  leave  $\tilde{\Gamma}$  invariant and form a group denoted by  $\Pi$ , which is isomorphic to  $P/Q$  by the natural projection  $\{\omega_r \mapsto \pi_r\}$ . As to  $\{u_r\}$ , they preserve the set  $\{-\vartheta, \alpha_i, i > 0\}$ . The relations  $\pi_r(\alpha_0) = \alpha_r = (u_r)^{-1}(-\vartheta)$  distinguish the indices  $r \in O'$ . Moreover (see e.g., [C4]):

$$(1.6) \quad \widehat{W} = \Pi \ltimes \tilde{W}, \quad \text{where } \pi_r s_i \pi_r^{-1} = s_j \quad \text{if } \pi_r(\alpha_i) = \alpha_j, \quad 0 \leq j \leq n.$$



**1.2. The length on  $\widehat{W}$ .** Setting  $\widehat{w} = \pi_r \widetilde{w} \in \widehat{W}$ ,  $\pi_r \in \Pi$ ,  $\widetilde{w} \in \widetilde{W}$ , the length  $l(\widehat{w})$  is by definition the length of the reduced decomposition  $\widetilde{w} = s_{i_1} \dots s_{i_2} s_{i_1}$  in terms of the simple reflections  $s_i$ ,  $0 \leq i \leq n$ . The number of  $s_i$  in this decomposition such that  $\nu_i = \nu$  is denoted by  $l_\nu(\widehat{w})$ .

The **length** can be also defined as the cardinality  $|\lambda(\widehat{w})|$  of the  $\lambda$ -set of  $\widehat{w}$ :

$$(1.7) \quad \lambda(\widehat{w}) \stackrel{\text{def}}{=} \widetilde{R}_+ \cap \widehat{w}^{-1}(\widetilde{R}_-) = \{\tilde{\alpha} \in \widetilde{R}_+, \widehat{w}(\tilde{\alpha}) \in \widetilde{R}_-\}, \quad \widehat{w} \in \widehat{W}.$$

Respectively,

$$(1.8) \quad \lambda(\widehat{w}) = \cup_\nu \lambda_\nu(\widehat{w}), \quad \lambda_\nu(\widehat{w}) \stackrel{\text{def}}{=} \{\tilde{\alpha} \in \lambda(\widehat{w}), \nu(\tilde{\alpha}) = \nu\}.$$

Note that  $\lambda(\widehat{w})$  is closed with respect to positive linear combinations. More exactly, if  $\tilde{\alpha} = u\tilde{\beta} + v\tilde{\gamma} \in \widetilde{R}$  for rational  $u, v > 0$ , then  $\tilde{\alpha} \in \lambda(\widehat{w})$  if  $\tilde{\beta} \in \lambda(\widehat{w}) \ni \tilde{\gamma}$ . Vice versa, if  $\lambda(\widehat{w}) \ni \tilde{\alpha} = u\tilde{\beta} + v\tilde{\gamma}$  for  $\tilde{\beta}, \tilde{\gamma} \in \widetilde{R}_+$  and rational  $u, v > 0$ , then either  $\tilde{\beta}$  or  $\tilde{\gamma}$  must belong to  $\lambda(\widehat{w})$ . Also,

$$(1.9) \quad \begin{aligned} &\tilde{\alpha} = [\alpha, \nu_\alpha j] \in \lambda(\widehat{w}) \Leftrightarrow \{[\alpha, \nu_\alpha i] \in \lambda(\widehat{w}) \\ &\text{for all } 0 \leq i \leq j \text{ where } i > 0 \text{ as } \alpha < 0\}. \end{aligned}$$

This property is obvious because  $\widehat{w}([\alpha, \nu_\alpha i]) = \widehat{w}(\tilde{\alpha}) + [0, \nu_\alpha(i - j)]$ .

The coincidence with the previous definition is based on the equivalence of the *length equality*

$$(1.10) \quad (a) \quad l_\nu(\widehat{w}\widehat{u}) = l_\nu(\widehat{w}) + l_\nu(\widehat{u}) \quad \text{for } \widehat{w}, \widehat{u} \in \widehat{W}$$

and the *cocycle relation*

$$(1.11) \quad (b) \quad \lambda_\nu(\widehat{w}\widehat{u}) = \lambda_\nu(\widehat{u}) \cup \widehat{u}^{-1}(\lambda_\nu(\widehat{w})),$$

which, in its turn, is equivalent to the *positivity condition*

$$(1.12) \quad (c) \quad \widehat{u}^{-1}(\lambda_\nu(\widehat{w})) \subset \widetilde{R}_+$$

and is also equivalent to the *embedding condition*

$$(1.13) \quad (d) \quad \lambda_\nu(\widehat{u}) \subset \lambda_\nu(\widehat{w}).$$

Formula (1.11) obviously includes the positivity condition (1.12). It also readily gives (d) and implies that

$$\lambda_\nu(\widehat{u}) \cap \widehat{u}^{-1}(\lambda_\nu(\widehat{w})) = \widehat{u}^{-1}(\widehat{u}(\lambda_\nu(\widehat{u})) \cap \lambda_\nu(\widehat{w})) = \emptyset$$

thanks to the general formula

$$\lambda_\nu(\widehat{w}^{-1}) = -\widehat{w}(\lambda_\nu(\widehat{w})).$$

Thus it results in the equality  $l_\nu(\widehat{w}\widehat{u}) = l_\nu(\widehat{w}) + l_\nu(\widehat{u})$  and we have the implications  $(a) \Leftarrow (b) \Rightarrow (c)$ .

The remaining implications  $(a) \Rightarrow (b) \Leftarrow (c)$  are based on the following simple general fact:

(1.14)

$$\lambda_\nu(\widehat{w}\widehat{u}) \setminus \{\lambda_\nu(\widehat{w}\widehat{u}) \cap \lambda_\nu(\widehat{u})\} = \widehat{u}^{-1}(\lambda_\nu(\widehat{w})) \cap \widetilde{R}_+ \text{ for any } \widehat{u}, \widehat{w}.$$

For instance, the length equality  $(a)$  readily implies  $(b)$  and  $(d)$  results in  $(c)$ . For the sake of completeness, let us deduce  $(b)$  from the positivity condition  $(c)$ . We follow [C4].

It suffices to check that  $\lambda_\nu(\widehat{w}\widehat{u}) \supset \lambda_\nu(\widehat{u})$ . If there exists a (positive)  $\widetilde{\alpha} \in \lambda_\nu(\widehat{u})$  such that  $(\widehat{w}\widehat{u})(\widetilde{\alpha}) \in \widetilde{R}_+$ , then

$$\widehat{w}(-\widehat{u}(\widetilde{\alpha})) \in \widetilde{R}_- \Rightarrow -\widehat{u}(\widetilde{\alpha}) \in \lambda_\nu(\widehat{w}) \Rightarrow -\widetilde{\alpha} \in \widehat{u}^{-1}(\lambda_\nu(\widehat{w})).$$

We come to a contradiction with the positivity. Hence  $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ .

Note that the embedding condition  $(d)$  readily gives the following well-known fact. Let  $R'$  be a root subsystem  $R' \subset \widetilde{R}$  with the simple roots that are simple in  $\widetilde{R}$  constituting a connected subset of the affine Dynkin diagram  $\widetilde{\Gamma}$ ,  $w'_0$  the greatest element in the corresponding Weyl group  $W'$ . Then

$$(1.15) \quad l(\widehat{w}) = l(w'_0\widehat{w}) + l(w'_0) \Leftrightarrow l(s_i\widehat{w}) = l(\widehat{w}) - 1 \text{ for all } \alpha_i \in R',$$

i.e.,  $\widehat{w}$  is divisible on the left by  $w'_0$  in the sense of reduced decompositions if and only if it is divisible by all  $s_i$  for  $\alpha_i \in R'$ . Indeed, it is equivalent to the “inverse” statement

$$l(\widehat{w}) = l(\widehat{w}w'_0) + l(w'_0) \Leftrightarrow l(\widehat{w}s_i) = l(\widehat{w}) - 1 \text{ for all } \alpha_i \in R',$$

that directly follows from  $(d)$  and the inclusion  $\lambda(w'_0) \subset \lambda(\widehat{w})$ . See also Theorem 2.1 below.

Applying (1.11) to the reduced decomposition  $\widehat{w} = \pi_r s_{i_l} \cdots s_{i_2} s_{i_1}$ ,

$$(1.16) \quad \lambda(\widehat{w}) = \{ \widetilde{\alpha}^l = \widetilde{w}^{-1} s_{i_l}(\alpha_{i_l}), \dots, \widetilde{\alpha}^3 = s_{i_1} s_{i_2}(\alpha_{i_3}), \widetilde{\alpha}^2 = s_{i_1}(\alpha_{i_2}), \widetilde{\alpha}^1 = \alpha_{i_1} \}.$$

It demonstrates directly that the cardinality  $l$  of the set  $\lambda(\widehat{w})$  equals  $l(\widehat{w})$ . Cf. [Hu], 4.5.

This set can be introduced for non-reduced decompositions as well. Let us denote it by  $\tilde{\lambda}(\hat{w})$  to differ from  $\lambda(\hat{w})$ . It always contains  $\lambda(\hat{w})$  and, moreover, can be represented in the form

$$(1.17) \quad \begin{aligned} \tilde{\lambda}(\hat{w}) &= \lambda(\hat{w}) \cup \tilde{\lambda}^+(\hat{w}) \cup -\tilde{\lambda}^+(\hat{w}), \\ \text{where } \tilde{\lambda}^+(\hat{w}) &= (\tilde{R}_+ \cap \tilde{\lambda}(\hat{w})) \setminus \lambda(\hat{w}). \end{aligned}$$

The coincidence with  $\lambda(\hat{w})$  is for reduced decompositions only.

Note that  $\tilde{\lambda}(\hat{w})$  depends on the choice of the decomposition and it is actually a *sequence*; the roots in (1.16) are ordered naturally. We will mainly treat  $\lambda(\hat{w})$  and  $\tilde{\lambda}(\hat{w})$  as *sequences* in this paper, for instance, when discussing the Bruhat ordering.

Let us consider the  $\lambda$ -sets of the reflections in  $\widetilde{W}$  and check another standard property of the  $\lambda$ -sets (see, [B] and [Hu], 4.6, Exchange Condition):

**Proposition 1.1.** *For  $\hat{w} \in \widehat{W}$ ,*

$$(1.18) \quad \lambda_\nu(\hat{w}) = \{\tilde{\alpha} > 0, l_\nu(\hat{w}s_{\tilde{\alpha}}) \leq l_\nu(\hat{w})\}.$$

*Proof.* It suffices to consider  $\lambda(\hat{w})$ . Also, we can the inequality here  $l(\hat{w}s_{\tilde{\alpha}}) < l(\hat{w})$ , since  $l(s_{\tilde{\alpha}})$  is odd (see below). Thanks to (1.16), the set  $\{\tilde{\alpha} > 0 \mid l(\hat{w}s_{\tilde{\alpha}}) < l(\hat{w})\}$  contains  $\lambda(\hat{w})$ . Obviously (1.18) holds for  $s_{\tilde{\alpha}} = s_{\alpha_i}$  ( $0 \leq i \leq n$ ).

An arbitrary element  $s_{\tilde{\alpha}}$  for  $\tilde{\alpha} > 0$  has a *reduced* decomposition in the form:

$$s_{\tilde{\alpha}} = s_{i_1}s_{i_2} \cdots s_{i_p}s_ms_{i_p} \cdots s_{i_2}s_{i_1}, \quad i_\bullet, m \geq 0.$$

Indeed,  $\alpha_i \in \lambda(s_{\tilde{\alpha}})$  if and only if  $(\tilde{\alpha}, \alpha_i) > 0$  because  $\{\alpha_i\}$  is a minimal positive affine root. Given such  $\alpha_i$ , the reflection  $s_{\tilde{\alpha}}$  is divisible by  $s_i$  on the right (i.e., has a reduced decomposition in the form  $\cdots s_i$ ) and  $l(s_{\tilde{\beta}}) \leq l(s_{\tilde{\alpha}})$  for  $\tilde{\beta} \stackrel{\text{def}}{=} s_i(\tilde{\alpha})$ . If  $l(s_{\tilde{\beta}}) = l(s_{\tilde{\alpha}})$ , then  $s_{\tilde{\beta}} = s_i s_{\tilde{\alpha}} s_i = s_{\tilde{\beta}}^{-1}$  is divisible by  $s_i$  on the left or on the right, which contradicts to  $(\tilde{\beta}, \alpha_i) = -(\tilde{\alpha}, \alpha_i) < 0$ . Therefore  $l(s_{\tilde{\beta}}) < l(s_{\tilde{\alpha}})$  and we can proceed by induction.

We have also obtained that the required decomposition can be started with an arbitrary simple  $\alpha_j \in \lambda(s_{\tilde{\alpha}})$  taken as  $\alpha_{i_1}$ . Moreover, the *sequence*  $\lambda(s_{\tilde{\alpha}})$  can be expressed as follows in terms of an (arbitrary) element  $\tilde{w} \in \widetilde{W}$  of minimal possible length such that  $\tilde{w}(\alpha) = \alpha_i$  for

some  $i \geq 0$ :

$$(1.19) \quad \lambda(s_{\tilde{\alpha}}) = \{s_{\tilde{\alpha}}(-\lambda(\tilde{w}))\}_{op} \cup \tilde{\alpha} \cup \lambda(\tilde{w}),$$

Given a sequence  $\{\cdot\}$ , by  $\{\cdot\}_{op}$ , we mean the inversion of its ordering.

If an arbitrary  $\tilde{w}$  sending  $\tilde{\alpha}$  to a simple root is taken here, then (1.19) holds for  $\lambda(s_{\tilde{\alpha}})$  considered as a *set* with possible cancelation of pairs of opposite roots as in (1.17).

Now, we can prove (1.18) by induction with respect to the length  $l(s_{\tilde{\alpha}})$  of the roots  $\tilde{\alpha}$  such that  $l(\hat{w}s_{\tilde{\alpha}}) < l(\hat{w})$ . As we already noticed, it holds for simple reflections. Generally, let  $\alpha_i \in \lambda(s_{\tilde{\alpha}})$ ,  $\tilde{\beta} \stackrel{\text{def}}{=} s_i(\tilde{\alpha})$ ,  $s_{\tilde{\beta}} = s_i s_{\tilde{\alpha}} s_i$ ,  $l(s_{\tilde{\beta}}) = l(s_{\tilde{\alpha}}) - 2$ .

If  $\alpha_i \notin \lambda(\hat{w})$ , then

$$l(\hat{w}s_{\tilde{\beta}}) = l((\hat{w}s_i)(s_i s_{\tilde{\alpha}} s_i)) < l(\hat{w}) + 1 = l(\hat{w}s_i),$$

and the induction statement gives that

$$\tilde{\beta} \in \lambda(\hat{w}s_i) \Rightarrow s_i(\tilde{\alpha}) \in s_i(\lambda(\hat{w})) \Rightarrow \tilde{\alpha} \in \lambda(\hat{w}).$$

Thus, we can assume that  $\alpha_i \in \lambda(\hat{w})$ , equivalently,  $\hat{w}(\alpha_i) < 0$ , equivalently,  $l(\hat{w}s_i) = l(\hat{w}) - 2$ . Since  $\alpha_i \in \lambda(s_{\tilde{\alpha}})$  and  $(\tilde{\alpha}, \alpha_i) > 0$ , we obtain:

$$(\hat{w}s_{\tilde{\alpha}})(\alpha_i) = \hat{w}(\alpha_i - 2 \frac{(\tilde{\alpha}, \alpha_i)}{(\tilde{\alpha}, \tilde{\alpha})} \tilde{\alpha}) < 0 \text{ provided } \hat{w}(\tilde{\alpha}) > 0.$$

If here  $\hat{w}(\tilde{\alpha}) < 0$ , then  $\tilde{\alpha} \in \lambda(\hat{w})$ , which gives the desired. Thus, it suffices to consider the case  $(\hat{w}s_{\tilde{\alpha}})(\alpha_i) < 0$ , where  $\alpha_i \in \lambda(\hat{w}s_{\tilde{\alpha}})$ .

In this case:

$$l((\hat{w}s_i)s_{\tilde{\beta}}) = l(\hat{w}s_{\tilde{\alpha}}s_i) \leq l(\hat{w}s_{\tilde{\alpha}}) - 1 \leq l(\hat{w}) - 2 = l(\hat{w}s_i).$$

By induction,  $\tilde{\beta} \in \lambda(\hat{w}s_i)$ ; therefore  $\tilde{\beta} \in s_i(\lambda(\hat{w}))$  and, finally,  $\tilde{\alpha} \in \lambda(\hat{w})$ .  $\square$

The following is an immediate corollary of (1.19):

$$(1.20) \quad \tilde{\alpha} \in \lambda(\hat{w}) \Leftrightarrow \lambda(s_{\tilde{\alpha}}) = \{\tilde{\beta}, -s_{\tilde{\alpha}}(\tilde{\beta}) \mid s_{\tilde{\alpha}}(\tilde{\beta}) \in \tilde{R}_-, \tilde{\beta} \in \lambda(\hat{w})\}.$$

In more detail, we have the following lemma.

**Lemma 1.2.** (i) Let  $\tilde{\alpha} = [\alpha, \nu_{\alpha}j] \in \tilde{R}_+$ . Then  $(\tilde{\alpha}, \tilde{\beta}) > 0$  for each  $\tilde{\beta} \in \lambda(s_{\tilde{\alpha}})$ . Given  $\tilde{\beta}, \tilde{\gamma} \in \lambda(s_{\tilde{\alpha}})$ , the sum  $\tilde{\beta} + \tilde{\gamma}$  belongs to  $\tilde{R}_+$  ( $\Rightarrow \tilde{\beta} + \tilde{\gamma} \in \lambda(s_{\tilde{\alpha}})$ ) if and only if  $\nu_{\alpha} \geq \nu_{\beta} = \nu_{\gamma}$  and  $\tilde{\beta} + \tilde{\gamma} = [\alpha, \nu_{\alpha}j']$ . In particular,  $(\tilde{\beta}, \tilde{\gamma}) \geq 0$  unless the latter condition holds.

(ii) Given a reduced decomposition of  $s_{\tilde{\alpha}}$ , its rank two Coxeter transformations of (consecutive) simple reflections are either  $s_i s_j s_i \mapsto s_j s_i s_j$  with the midpoints corresponding to  $[\alpha, \nu_{\alpha} j'] \in \lambda(s_{\tilde{\alpha}})$  or  $s_i s_j \mapsto s_j s_i$  otherwise.

(iii) Let  $\tilde{\beta}$  be the first root in  $\lambda(s_{\tilde{\alpha}})$  (then it must be simple). If  $\tilde{\beta}$  can be made a neighbor of  $\tilde{\alpha}$  in  $\lambda(\tilde{\alpha})$  using the Coxeter transforms, then there exists a reduced decomposition of  $s_{\tilde{\alpha}}$  such that  $\tilde{\beta}$  is the first root in  $\lambda(s_{\tilde{\alpha}})$  and all roots  $\tilde{\gamma}$  between  $\tilde{\beta}$  and  $\tilde{\alpha}$  are orthogonal to  $\tilde{\beta}$  unless they satisfy  $s_{\tilde{\beta}}(\tilde{\gamma}) = \tilde{\beta} + \tilde{\gamma} = [\alpha, \nu_{\alpha} j']$ , for some  $j' < j$ .

*Proof.* The positivity  $(\tilde{\alpha}, \tilde{\beta}) > 0$  is necessary (generally, not sufficient) for  $\tilde{\beta}$  to belong to  $\lambda(s_{\tilde{\alpha}})$ . If  $(\tilde{\beta}, \tilde{\gamma}) < 0$ , then  $s_{\tilde{\beta}}(\tilde{\gamma}) = \tilde{\gamma} + \tilde{\beta} \in \lambda(\tilde{w})$  assuming that  $\nu_{\beta} \geq \nu_{\gamma}$ . One has:

$$s_{\tilde{\alpha}}(\tilde{\beta} + \tilde{\gamma}) = \tilde{\beta} + \tilde{\gamma} - \frac{(\tilde{\alpha}, \tilde{\beta})}{\nu_{\alpha}} \tilde{\alpha} - \frac{(\tilde{\alpha}, \tilde{\gamma})}{\nu_{\alpha}} \tilde{\alpha}.$$

However, the coefficient of  $\tilde{\alpha}$  must be  $-(\tilde{\alpha}, \tilde{\gamma})/\nu_{\alpha}$  unless the nonaffine components of  $\tilde{\beta} + \tilde{\gamma}$  and  $\tilde{\alpha}$  coincide and  $\nu_{\alpha} \geq \nu_{\beta} = \nu_{\gamma}$ ; we use that  $s_{\tilde{\beta}}(\tilde{\gamma})$  has the same length as  $\tilde{\gamma}$ .

Claim (ii) follows from (i) because any sequence of roots corresponding to a rank two Coxeter transformation contains a pair of roots with the negative scalar product. In (iii), we move  $\tilde{\beta}$  to the position next to  $\tilde{\alpha}$ ; all Coxeter transformations we use must satisfy (ii). Then we can move  $\tilde{\beta}$  (back) to its first position (it is a simple root) using the Coxeter transforms in the segment of  $\lambda(s_{\tilde{\alpha}})$  before  $\tilde{\alpha}$ .  $\square$

The sequence  $\lambda(\hat{w}) = \{\tilde{\alpha}^l, \dots, \tilde{\alpha}^1\}$ , where  $l = l(\tilde{w})$ , and the element  $\pi_r$  determine  $\hat{w} \in \widehat{W}$  uniquely:

$$\begin{aligned} \alpha_{i_1} &= \tilde{\alpha}^1, \alpha_{i_2} = s^1(\tilde{\alpha}^2), \dots, \alpha_{i_p} = s^1 s^2 \dots s^{p-1}(\tilde{\alpha}^p), \dots \\ \alpha_{i_l} &= s^1 s^2 \dots s^{l-1}(\tilde{\alpha}^l), \text{ where} \\ (1.21) \quad s^p &= s_{\tilde{\alpha}^p}, \hat{w} = \pi_r s_{i_l} \dots s_{i_1} = \pi_r s^1 \dots s^l. \end{aligned}$$

Notice the order of the reflections  $s^p$  in the decomposition of  $\hat{w}$  is *inverse*.

A stronger fact holds. The  $\lambda(\hat{w})$  considered as an unordered set determines  $\hat{w}$  uniquely up to  $\pi_r \in \Pi$  (on the left). This statement can be readily checked by induction with respect to  $l = l(\hat{w})$ . Indeed, there exists at least one simple  $\alpha_i \in \lambda(\hat{w})$  and *any* such  $\alpha_i$  can be made

$\alpha_{i_1}$ ; this means that  $l(\widehat{w}') = l - 1$  for  $\widehat{w}' = \widehat{w}s_i$ . Therefore the set  $s_i(\lambda(\widehat{w}) \setminus \alpha_i)$  equals  $\lambda(\widehat{w}')$  and determines  $\widehat{w}'$  uniquely by the induction statement for  $l - 1$ .

**1.3. Reduction modulo  $W$ .** It generalizes the construction of the elements  $\pi_b$  for  $b \in P_+$ . As a matter of fact, the reduction modulo  $W$  is a *formal* particular case of a more general construction of the elements  $\widehat{w}^\circ$  from Section 4 below when  $\widetilde{R}^0 = R$ . This can be used to obtain almost all claims of the following proposition, that is from [C6].

**Proposition 1.3.** *Given  $b \in P$ , there exists a unique decomposition  $b = \pi_b u_b$ ,  $u_b \in W$  satisfying one of the following equivalent conditions:*

- (i)  $l(\pi_b) + l(u_b) = l(b)$  and  $l(u_b)$  is the greatest possible,
- (ii)  $\lambda(\pi_b) \cap R = \emptyset$ .

*The latter condition implies that  $l(\pi_b) + l(w) = l(\pi_b w)$  for any  $w \in W$ . Besides, the relation  $u_b(b) \stackrel{\text{def}}{=} b_- \in P_- = -P_+$  holds, which, in its turn, determines  $u_b$  uniquely if one of the following equivalent conditions is imposed:*

- (iii)  $l(u_b)$  is the smallest possible,
- (iv) if  $\alpha \in \lambda(u_b)$  then  $(\alpha, b) \neq 0$ .

□

Note that the relation  $l(\pi_b) + l(w) = l(\pi_b w)$  for any  $w \in W$  is a special property of  $\widetilde{R}^0 = R$  ( $\widetilde{R}^0$  is from Section 4.1). Generally,  $l(\widehat{w}^\circ \widehat{u}) \neq l(\widehat{w}^\circ) + l(\widehat{u})$  for  $\widehat{u} \in \widehat{W}^0$ , where  $\lambda(\widehat{w}^\circ) \cap \widetilde{R}^0 = \emptyset$ .

Condition (ii) readily gives a complete description of the set  $\pi_P = \{\pi_b, b \in P\}$ , namely, only  $[\alpha < 0, \nu_\alpha j > 0]$  can appear in  $\lambda(\pi_b)$  due to (1.9); see also (1.25) below.

We note the following application of Theorem 2.1 below. A *sequence*

$$\lambda \subset \widetilde{R}_+[-] \stackrel{\text{def}}{=} \{[\alpha, \nu_\alpha j], \alpha \in R_-, j > 0\}$$

is in the form  $\lambda(\pi_b)$  when and only when the following three conditions hold:

- (i) assuming that  $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma} = \widetilde{\alpha} + \widetilde{\beta} \in \widetilde{R}_+[-]$ , if  $\widetilde{\alpha}, \widetilde{\beta} \in \lambda$  then  $\widetilde{\gamma} \in \lambda$  and  $\widetilde{\gamma}$  appears between  $\widetilde{\alpha}, \widetilde{\beta}$ ; if  $\widetilde{\alpha} \notin \lambda$  then  $\widetilde{\beta}$  belongs to  $\lambda$  and appears in  $\lambda$  before  $\widetilde{\gamma}$ ;
- (ii) if  $\widetilde{\alpha} = [\alpha, \nu_\alpha j] \in \lambda$  then  $[\alpha, \nu_\alpha j'] \in \lambda$  as  $j > j' > 0$  and it appears in  $\lambda$  before  $\widetilde{\alpha}$ ;

(iii) if  $\tilde{\beta} \in \lambda$  and  $\tilde{\gamma} = \tilde{\beta} - [\alpha, \nu_\alpha j] \in \tilde{R}_+[-]$  for  $\alpha \in R_+$ ,  $j \geq 0$  then  $\tilde{\gamma} \in \lambda$  and it appears before  $\tilde{\beta}$ .

If  $\lambda$  is treated as an unordered set, then the existence of a representation  $\lambda = \lambda(\pi_b)$  for some  $b$  is equivalent to (i+ii+iii) without the claims concerning the ordering due to (1.25)

Since  $\pi_b = bu_b^{-1} = u_b^{-1}b_-$ , the set  $\pi_P$  can be described explicitly in terms of  $P_-$ :

$$(1.22) \quad \pi_P = \{ u^{-1}b_- \text{ for } b_- \in P_-, u \in W \text{ such that } \alpha \in \lambda(u^{-1}) \Rightarrow (\alpha, b_-) \neq 0 \}.$$

Indeed,  $\lambda(u^{-1}b_-) = (-b_-)(\lambda(u^{-1})) \cup \lambda(b_-)$  and (1.22) is necessary and sufficient for  $\lambda(u^{-1}b_-) \cap R = \emptyset$ . Note that using  $\lambda(u_b^{-1}) = -u_b(\lambda(u_b))$ ,

$$(1.23) \quad \alpha \in \lambda(u_b^{-1}) \Rightarrow (u_b^{-1}(\alpha), b) = (\alpha, u_b(b)) = (\alpha, b_-) \neq 0.$$

Actually, it suffices to check (1.22) for simple roots  $\alpha_i \in \lambda(u^{-1})$  only.

Using the longest element  $w'_0$  in the centralizer  $W'_0$  of  $b_-$ , the corresponding  $u$  that can be taken constitute the set

$$\{u \mid l(u^{-1}w'_0) = l(w'_0) + l(u^{-1})\}.$$

Their number is  $|W|/|W'_0|$ .

Note that Proposition 1.3,(ii) gives that any partial product  $\hat{u} = s_{i_p} \cdots s_{i_1}$  for a reduced decomposition  $\pi_b = \pi_r s_{i_l} \cdots s_{i_1}$  belongs to  $\pi_P$ . Equivalently, if  $\hat{u}$  satisfies  $l(\pi_b) = l(\hat{u}) + l(\pi_b \hat{u}^{-1})$ , then  $\hat{u} = \pi_c$  for  $c = \hat{u}((0))$  in the notation from (1.29).

For  $\tilde{\alpha} = [\alpha, \nu_\alpha j] \in \tilde{R}_+$ , one has:

$$(1.24) \quad \lambda(b) = \{\tilde{\alpha}, (b, \alpha^\vee) > j \geq 0 \text{ if } \alpha \in R_+, (b, \alpha^\vee) \geq j > 0 \text{ if } \alpha \in R_-\},$$

$$(1.25) \quad \lambda(\pi_b) = \{\tilde{\alpha}, \alpha \in R_-, (b_-, \alpha^\vee) > j > 0 \text{ if } u_b^{-1}(\alpha) \in R_+, (b_-, \alpha^\vee) \geq j > 0 \text{ if } u_b^{-1}(\alpha) \in R_-\},$$

$$(1.26) \quad \lambda(\pi_b^{-1}) = \{\tilde{\alpha} \in \tilde{R}_+, -(b, \alpha^\vee) > j \geq 0\},$$

$$(1.27) \quad \lambda(u_b) = \{\alpha \in R_+, (b, \alpha^\vee) > 0\}.$$

For instance,  $l(b) = l(b_-) = -2(\rho^\vee, b_-)$  for  $2\rho^\vee = \sum_{\alpha > 0} \alpha^\vee$ . Note that nonempty  $\lambda(\pi_b^{-1})$  always contain nonaffine roots; thus  $\pi_b^{-1}$  can be represented in the form  $\pi_c$  if and only if  $\pi_b = \pi_r$  for some  $r \in O$ :  $\pi_P \cap \pi_P^{-1} = \Pi$ .

Let us demonstrate how this formalism works for calculating the set  $\lambda(s_{\tilde{\beta}})$  for  $\tilde{\beta} = [-\beta, l\nu_{\beta}]$ , where  $\beta \in R_+, l > 0$ . This set can be determined directly from the definition of  $\lambda(s_{\tilde{\beta}})$ , but it is useful to establish a connection with  $(-l\beta)' = s_{\beta}s_{\tilde{\beta}}$ . One has:

$$\begin{aligned}\lambda(s_{\beta}) &= s_{\beta}(R_-) \cap R_+, \quad s_{\tilde{\beta}}(\lambda(s_{\beta})) = \{[-\alpha, (\beta, \alpha)l], \alpha \in \lambda(s_{\beta})\}, \\ \text{indeed, } s_{\tilde{\beta}}(\alpha) &= \tilde{\alpha} + (\alpha, \beta^{\vee})\tilde{\beta} = s_{\beta}(\alpha) + [0, (\alpha, \beta)l] \\ &= -\alpha' + [0, (\alpha', \beta)l] \quad \text{for } \alpha' = -s_{\beta}(\alpha) \in -s_{\beta}(\lambda(s_{\beta})) = \lambda(s_{\beta}).\end{aligned}$$

Here  $(\beta, \alpha)$  must be greater than zero. Therefore the above decomposition of  $-l\beta \in Q$  is *reduced* thanks to the positivity condition (c) above, and we can apply (1.24).

Note that the inequality  $(\beta, \alpha) > 0$  is necessary for  $\alpha > 0$  to belong to  $\lambda(s_{\beta})$  but not sufficient; for  $\beta = \vartheta$  it is sufficient.

Using (1.24),

$$\begin{aligned}\lambda(-l\beta) &= \{\tilde{\alpha} = [-\alpha, \nu_{\alpha}j] \mid (\alpha^{\vee}, \beta)l > j \geq 0 \text{ as } \alpha \in R_-, \\ &\quad (\alpha^{\vee}, \beta)l \geq j > 0 \text{ as } \alpha \in R_+\}.\end{aligned}$$

Finally,

$$\begin{aligned}\lambda(s_{\tilde{\beta}}) &= \{[-\alpha, \nu_{\alpha}j] \in \tilde{R}_+ \mid (\alpha^{\vee}, \beta)l > j \geq 0\} \cup \\ (1.28) \quad &\{[-\alpha, (\beta, \alpha)l] \mid \alpha > 0 < s_{\beta}(\alpha), (\beta, \alpha) > 0\}.\end{aligned}$$

For example,  $\lambda(s_{\tilde{\beta}}) \cap R_+ = \{\alpha' > 0 \mid (\alpha', \beta) < 0\}$ ; this set is never empty unless  $(\beta, R_+) \geq 0$ ; the latter occurs, for instance, for  $\beta = \vartheta$ .

Let us show explicitly how the transformation  $-s_{\tilde{\beta}}$  acts in  $\lambda(s_{\tilde{\beta}})$ :

(a) the subset  $\{[-\alpha, (\beta, \alpha)l] \mid \alpha > 0 < s_{\beta}(\alpha), (\beta, \alpha) > 0\}$  maps exactly to  $\lambda(s_{\tilde{\beta}}) \cap R_+ = \{[\alpha', 0] \mid \alpha' > 0, (\alpha', \beta) < 0\}$ ,

(b) the subset  $\{[-\alpha, \nu_{\alpha}j] \in \tilde{R}_+ \mid (\alpha^{\vee}, \beta)l > j > 0, \alpha > 0 < s_{\beta}(\alpha)\}$  maps onto  $\{[-\alpha', \nu_{\alpha'}j] \in \tilde{R}_+ \mid ((\alpha')^{\vee}, \beta)l > j > 0, \alpha' < 0 > s_{\beta}(\alpha')\}$ ,

(c) the subset  $\{[-\alpha, \nu_{\alpha}j] \in \tilde{R}_+ \mid (\alpha^{\vee}, \beta)l > j > 0, \alpha > 0 > s_{\beta}(\alpha)\}$  remains invariant under the action of  $-s_{\tilde{\beta}}$ .

We will need later the following **affine action** of  $\widehat{W}$  on  $z \in \mathbb{R}^n$ :

$$\begin{aligned}(wb)((z)) &= w(b+z), \quad w \in W, b \in P, \\ (1.29) \quad s_{\tilde{\alpha}}((z)) &= z - ((z, \alpha^{\vee}) + j)\alpha, \quad \tilde{\alpha} = [\alpha, \nu_{\alpha}j] \in \tilde{R}.\end{aligned}$$



For instance,  $(bw)((0)) = b$  for any  $w \in W$ . The relation to the above action is given in terms of the **affine pairing**  $([z, l], z' + d) \stackrel{\text{def}}{=} (z, z') + l :$

$$(1.30) \quad (\widehat{w}([z, l]), \widehat{w}([z', l]) + d) = ([z, l], z' + d) \text{ for } \widehat{w} \in \widehat{W},$$

where we treat  $(\cdot, \cdot + d)$  formally (one can add  $d$  to  $\mathbb{R}^{n+1}$  and extend  $(\cdot, \cdot)$  correspondingly).

Introducing the **basic affine Weyl chamber**

$$\mathfrak{C}^a = \bigcap_{i=0}^n \mathfrak{L}_{\alpha_i}, \quad \mathfrak{L}_{[\alpha, \nu_\alpha j]} = \{z \in \mathbb{R}^n, (z, \alpha) + j > 0\},$$

we come to another interpretation of the  $\lambda$ -sets:

$$(1.31) \quad \lambda_\nu(\widehat{w}) = \{\tilde{\alpha} \in \tilde{R}_+, \mathfrak{C}^a \not\subset \widehat{w}(\mathfrak{L}_{\tilde{\alpha}}), \nu_\alpha = \nu\}.$$

Equivalently, taking a vector  $\xi \in \mathfrak{C}^a$ ,

$$(1.32) \quad \lambda(\widehat{w}) = \{\tilde{\alpha} \in \tilde{R} \mid (\tilde{\alpha}^\vee, \xi + d) > 0 > (\tilde{\alpha}^\vee, \xi' + d)\}$$

for  $\xi' \in \widehat{w}^{-1}(\mathfrak{C}^a)$ . Thus, we come to the following geometric description of the  $\lambda$ -sets (cf. Theorem 2.1, (a, b) below).

**Proposition 1.4.** *The  $\lambda$ -sets are exactly those in the form (1.32) for an arbitrary  $\xi \in \mathfrak{C}^a$  and an arbitrary vector  $\xi'$  inside a certain (affine) Weyl chamber, i.e., provided that  $(\xi', \alpha^\vee) \notin \nu_\alpha \mathbb{Z}$  for all  $\alpha \in R$ . Given a generic segment in  $\mathbb{R}^n$  from  $\xi$  to  $\xi'$ , its consecutive intersections with the affine root hyperplanes arrange such set into a  $\lambda$ -sequence.  $\square$*

Geometrically,  $\Pi$  is the group of all elements of  $\widehat{W}$  preserving  $\mathfrak{C}^a$  with respect to the affine action. Similarly, the elements  $\pi_b^{-1}$  for  $b \in P$  are exactly those sending  $\mathfrak{C}^a$  to the basic nonaffine Weyl chamber  $\mathfrak{C} \stackrel{\text{def}}{=} \{z \in \mathbb{R}^n, (z, \alpha_i) > 0 \text{ as } i > 0\}$ . More generally, given two finite sets of positive affine roots  $\{\tilde{\beta} = [\beta, \nu_\beta i]\}$  and  $\{\tilde{\gamma} = [\gamma, \nu_\gamma j]\}$ , the closure of the union of  $\widehat{w}^{-1}(\mathfrak{C}^a)$  over  $\widehat{w} \in \widehat{W}$  such that  $\tilde{\beta} \notin \lambda(\widehat{w}) \ni \tilde{\gamma}$  equals

$$\{z \in \mathbb{R}^n, (z, \beta) + i \geq 0, (z, \gamma) + j \leq 0 \text{ for all } \tilde{\beta}, \tilde{\gamma}\}.$$

**Comment.** Recall that for an arbitrary  $z \in \mathbb{R}^n$ , there exists a unique  $z_* = \tilde{w}([z]) \in \overline{\mathfrak{C}^a}$ , where  $\tilde{w} \in \widehat{W}$ ,  $\overline{\mathfrak{C}^a}$  is the closure of  $\mathfrak{C}^a$  in  $\mathbb{R}^n$ , and the centralizer of  $z_*$  in  $\widehat{W}$  is generated by simple reflections. For instance,  $a_* = 0$  for  $a \in Q$ ,  $b_* \in \{\omega_r, r \in O\}$  for  $b \in P$ .

The (classical) construction is as follows. We draw a line from a small deformation  $z^\epsilon$  of  $z$  to an arbitrary vector in  $\mathfrak{C}^a$ . It goes through a chain of affine chambers and readily gives a reduced decomposition for  $\tilde{w}$  sending  $z^\epsilon$  to the required  $z_* = \tilde{w}((z)) \in \mathfrak{C}^a$ .

If  $\tilde{u}((z_*)) = z_* \in \overline{\mathfrak{C}}^a$ , then  $\tilde{u}((z_*^\epsilon))$  remains in a small neighborhood of  $z_*$  for a small deformation  $z_*^\epsilon \in \mathfrak{C}^a$  of  $z_*$ . Considering the line from  $z_*^\epsilon$  to  $\tilde{u}((z_*^\epsilon))$ , we obtained a decomposition of  $\tilde{u}$ , where all simple reflections are in the hyperplanes sufficiently close to  $z_*$ . Thus all these hyperplanes are actually through  $z_*$  and, therefore, contain the *face* through  $z_*$ :  $\{z \mid (\tilde{\alpha}^\vee, z_* + d) = 0 \Rightarrow (\tilde{\alpha}^\vee, z + d) = 0 \text{ for } \tilde{\alpha} \in \tilde{R}\}$ .  $\square$

**Lemma 1.5.** *For an arbitrary  $b \in P$ , if  $\pi_r((b)) = b$  for  $\pi_r \in \Pi$ , then  $r = 0$ , that is,  $\pi_r = \text{id}$ .*

*Proof.* If  $\pi_r((b)) = b$  then  $((-b)\pi_r b)((0)) = 0$ . The element  $(-b)\pi_r b$  for  $r \neq 0$  does not belong to  $\widehat{W}$ , which is normal in  $\widehat{W}$ . However,  $\widehat{w}((0)) \Rightarrow \widehat{w} \in W$ .

It is instructional to check this claim explicitly. Introducing the map  $i \mapsto \widehat{i}$  by  $\alpha_{\widehat{i}} = \pi_r(\alpha_i)$ ,  $\pi_r((b)) = b$  for  $r \neq 0$  is equivalent to

$$b = \sum_{j=1}^n n_j \omega_j, \quad n_{\widehat{i}} = n_i \in \mathbb{Z} \quad \text{for} \quad n_0 \stackrel{\text{def}}{=} 1 - (\vartheta, b), \quad 0 \leq i \leq n.$$

Indeed,  $\pi_r((b)) = b \Leftrightarrow (\alpha_i^\vee, b + d) = (\alpha_{\widehat{i}}^\vee, b + d)$ , that is

$$\begin{aligned} (\alpha_{r'}^\vee, b) &= 1 - (\vartheta, b) = (\alpha_r^\vee, b), \quad \text{where } \widehat{r'} = 0, \\ (\alpha_j^\vee, b) &= (\alpha_{\widehat{j}}^\vee, b) \quad \text{for } 0 \neq j \neq r'. \end{aligned}$$

Setting  $\vartheta = \sum_j \kappa_j \alpha_j^\vee$  and using that  $\kappa_r = 1 = \kappa_{r'}$ ,

$$(1.33) \quad b = \sum_{j=1}^n n_j \omega_j = \pi_r((b)) \in P \Leftrightarrow$$

$$\left\{ n_{\widehat{j}} = n_j \in \mathbb{Z} \quad \text{for } 0 \neq j \neq r', \quad n_r = n_{r'}, \quad n_r + \sum_{j=1}^n \kappa_j n_j = 1 \right\}.$$

The sum  $n_r + \sum_j \kappa_j n_j$  is divisible by the order of  $\pi_r$  in the case of  $A_n$ , so there are no such  $b$  for this root system. Similarly, they do not exist in the case of  $B_{2l+1}$  ( $l > 1$ ) and for  $\pi_r$  of order 4 ( $D_{2l+1}$ , ( $l > 2$ )); indeed,  $\pi_r$  has no fixed nonaffine  $\alpha_j^\vee$  in these cases and the sum is divisible by 2. In the remaining cases,  $\pi_r \neq \text{id}$  have fixed nonaffine

$\alpha_j^\vee$ ; the corresponding labels  $\kappa_j$  are 2 for  $B, C, D$  and  $3 = |\Pi|$  for  $E_6$ , therefore, the sum is divisible by 2 or 3 and (1.33) is impossible.  $\square$

The element  $b_- = u_b(b)$  is a unique element from  $P_-$  that belongs to the orbit  $W(b)$ . Thus the equality  $c_- = b_-$  means that  $b, c$  belong to the same orbit. We will also use  $b_+ \stackrel{\text{def}}{=} w_0(b_-)$ , a unique element in  $W(b) \cap P_+$ . In terms of the elements  $\pi_b$ ,

$$u_b \pi_b = b_-, \pi_b^{-1} u_b^{-1} = b_+^s, b^s = -w_0(b).$$

Note that  $l(\pi_b w) = l(\pi_b) + l(w)$  for all  $b \in P, w \in W$ . For instance,

$$(1.34) \quad \begin{aligned} l(b_- w) &= l(b_-) + l(w), \quad l(w b_+) = l(b_+) + l(w), \\ l(u_b \pi_b w) &= l(u_b) + l(\pi_b) + l(w) \quad \text{for } b \in P, w \in W. \end{aligned}$$

**1.4. Partial ordering on  $P$ .** It is necessary in the theory of non-symmetric polynomials. See [O3, M5]. This ordering was also used in [C4] in the process of calculating the coefficients of  $Y$ -operators. The definition is as follows:

$$(1.35) \quad b \leq c, c \geq b \quad \text{for } b, c \in P \quad \text{if } c - b \in Q_+,$$

$$(1.36) \quad b \preceq c, c \succeq b \quad \text{if } b_- < c_- \text{ or } \{b_- = c_- \text{ and } b \leq c\}.$$

Recall that  $b_- = c_-$  means that  $b, c$  belong to the same  $W$ -orbit. We write  $<, >, \prec, \succ$  respectively if  $b \neq c$ .

The following sets

$$(1.37) \quad \begin{aligned} \sigma(b) &\stackrel{\text{def}}{=} \{c \in P, c \succeq b\}, \quad \sigma_*(b) \stackrel{\text{def}}{=} \{c \in P, c \succ b\}, \\ \sigma_-(b) &\stackrel{\text{def}}{=} \sigma(b_-), \quad \sigma_+(b) \stackrel{\text{def}}{=} \sigma_*(b_+) = \{c \in P, c_- > b_-\}. \end{aligned}$$

are convex. By **convex**, we mean that if  $c, d = c + r\alpha \in \sigma$  for  $\alpha \in R_+, r \in \mathbb{Z}_+$ , then

$$(1.38) \quad \{c, c + \alpha, \dots, c + (r-1)\alpha, d\} \subset \sigma.$$

The convexity of the intersections  $\sigma(b) \cap W(b), \sigma_*(b) \cap W(b)$  is by construction. For the sake of completeness, let us check the convexity of the sets  $\sigma_\pm(b)$ .

Both sets are  $W$ -invariant. Indeed,  $c_- > b_-$  if and only if  $b_+ > w(c) > b_-$  for all  $w \in W$ . The set  $\sigma_-(b)$  is the union of  $\sigma_+$  and the orbit  $W(b)$ . Here we use that  $b_+$  and  $b_-$  are the greatest and the least elements of  $W(b)$  with respect to " $>$ ". This is known (and can be

readily checked by the induction with respect to the length - see e.g., [C4]).

If the endpoints of (1.38) are between  $b_+$  and  $b_-$ , then it is true for the orbits of all inner points even if  $w \in W$  changes the sign of  $\alpha$  (and the order of the endpoints). Also the elements from  $\sigma(b)$  strictly between  $c$  and  $d$  (i.e.,  $c + q\alpha$ ,  $0 < q < r$ ) belong to  $\sigma_+(b)$ . This gives the required.

The next propositions are essentially from [C7]. We will use the standard Bruhat ordering. Given  $\hat{w} \in \widehat{W}$ , the *standard Bruhat set*  $\mathcal{B}(\hat{w})$  is formed by  $\hat{u}$  obtained by striking out any number of  $\{s_j\}$  from a reduced decomposition of  $\hat{w} \in \widehat{W}$ . The set  $\mathcal{B}(\hat{w})$  does not depend on the choice of the reduced decompositions.

**Proposition 1.6.** (i) Let  $c = \hat{u}((0))$ ,  $b = \hat{w}((0))$  and  $\hat{u} \in \mathcal{B}(\hat{w})$ . The latter means that  $\hat{u}$  can be obtained by deleting simple reflections from any reduced decomposition of  $\hat{w}$ , say, from the product of the reduced decompositions of  $\pi_b$  and  $w$  in the decomposition  $\hat{w} = \pi_b w$ . Then  $c \succeq b$  and  $b - c$  is a linear combination of the non-affine components of the corresponding roots from  $\lambda(\hat{w}^{-1})$ ; also,  $c = b$  if and only if  $s_j$  are deleted only from the reduced decomposition of  $w$ .

(ii) Letting  $b = s_i((c))$  for  $0 \leq i \leq n$ , if the element  $s_i \pi_c$  belongs to  $\{\pi_a, a \in P\}$  then it equals  $\pi_b$ . It happens if and only if  $(\alpha_i, c + d) \neq 0$ . More precisely, the following three conditions are equivalent:

$$(1.39) \quad \{c \succ b\} \Leftrightarrow \{(\alpha_i^\vee, c + d) > 0\} \Leftrightarrow \{s_i \pi_c = \pi_b, l(\pi_b) = l(\pi_c) + 1\}.$$

If the latter holds, then  $\lambda(\pi_b) = \pi_c^{-1}(\alpha_i) \cap \lambda(\pi_c)$ , where

$$\pi_c^{-1}(\alpha_i) = u_c(\alpha_i) + [0, (c, \alpha_i)].$$

□

The following lemma from [C6] completes (ii); it describes the case  $(\alpha_i^\vee, c + d) = 0$ .

**Lemma 1.7.** The condition  $(\alpha_i, c + d) = 0$  for  $0 \leq i \leq n$  equivalently, the condition  $(\alpha_i, b + d) = 0$ , implies that  $u_c(\alpha_i) = \alpha_j$  as  $i > 0$  or  $u_c(-\vartheta) = \alpha_j$  as  $i = 0$  for a proper index  $j > 0$ . Given  $c$  and  $i$ , the existence of such  $\alpha_j$  and the equality  $(\alpha_j, c_-) = 0$  are equivalent to  $(\alpha_i, c + d) = 0$ .

*Proof.* If  $i > 0$  then  $\alpha \stackrel{\text{def}}{=} u_c(\alpha_i) > 0$  and  $(\alpha, c_-) = 0$ . If  $\alpha = \beta + \gamma$  for positive roots  $\beta, \gamma$ , then  $(\beta, c_-) = 0 = (\gamma, c_-)$ . Hence  $\beta' = u_c^{-1}(\beta) > 0$ ,

$\gamma' = u_c^{-1}(\gamma) > 0$  and therefore  $\alpha_i = \beta + \gamma$ , which is impossible. Thus  $\alpha$  must be simple.

Similarly,  $(\vartheta, c) = 1$  implies that  $\alpha \stackrel{\text{def}}{=} u_c(-\vartheta) > 0, (\alpha, b_-) = -1$ . Let  $\alpha = \beta + \gamma$ , where  $\beta > 0 < \gamma$ . Since  $\vartheta$  and  $\alpha$  are short, we can assume that at least one of them is short, but it will follow automatically. Then, transposing  $\beta \leftrightarrow \gamma$  if necessary we obtain that  $(\beta, c_-) = -1, (\gamma, c_-) = 0$ . Since  $(\beta, c_-) = \nu_\beta(\beta^\vee, c_-)$  we conclude that  $\nu_\beta = 1$  and  $\beta$  is short. Hence  $\beta' = -u_c^{-1}(\beta) > 0, \gamma' = -u_c^{-1}(\gamma) < 0$  and therefore  $\vartheta = \beta' + \gamma' < \beta'$ . The latter results in  $\vartheta^\vee < (\beta')^\vee$ , which is impossible since  $\vartheta^\vee = \vartheta$  is the maximal positive root in  $R_+^\vee$ .  $\square$

Combining the lemma with (1.25), we come to the following corollary:

**Corollary 1.8.** *Let us take  $c \in P$  such that  $\lambda(\pi_c)$  contains  $[-\alpha, \nu_\alpha j]$  for each  $\alpha \in R_+$  where  $j > 0$  (then it holds for  $j = 1$ ). It results in  $-(c_-, \alpha_{i'}) > 0$  for all  $n \geq i' > 0$  and, given  $\widehat{w} \in \widehat{W}$ , the condition  $l(\widehat{w}) + l(\pi_c) = l(\widehat{w}\pi_c)$  implies that  $\widehat{w}\pi_c = \pi_b$  for some  $b \in P$ , having the same property as  $c$  (due to  $\lambda(\pi_c) \subset \lambda(\pi_b)$ ).*

*Proof.* Here the condition  $-(c_-, \alpha_{i'}) > 0$  for all  $i' > 0$  (we can write  $c_- \in P_-$ ) guarantees that  $s_i\pi_c$  is in the form  $\pi_b$  for any  $i \geq 0$ . This condition follows from the assumption for  $c$  (but is somewhat weaker). Provided  $l(s_i\pi_c) = l(\pi_c) + 1$ , we obtain that  $\lambda(s_i\pi_c) = \lambda(\pi_b) \supset \lambda(\pi_c)$  and can continue by induction with respect to  $l(\pi_c)$ .

See also the next proposition, that gives an explicit description of the changes of the  $\lambda$ -sets  $\lambda(\pi_c)$  and  $c_-$  upon the multiplication by simple reflections.  $\square$

**Proposition 1.9.** (i) *Assuming (1.39), let  $i > 0$ . Then  $b = s_i(c)$ ,  $b_- = c_-$  and  $u_b = u_c s_i$ . The set  $\lambda(\pi_b)$  is obtained from  $\lambda(\pi_c)$  by adding  $[\alpha, (c_-, \alpha)]$  for  $\alpha = u_c(\alpha_i)$ , i.e., by replacing the inequality  $(c_-, \alpha^\vee) > j > 0$  in (1.25) with  $(c_-, \alpha^\vee) \geq j > 0$ . Here  $\alpha \in R_-$  and  $(c_-, \alpha^\vee) = (c, \alpha_i^\vee) > 0$ .*

(ii) *In the case  $i = 0$ , the following holds:  $b = \vartheta + s_\vartheta(c)$ , the element  $c$  is from  $\sigma_+(b)$ ,  $b_- = c_- - u_c(\vartheta) \in P_-$  and  $u_b = u_c s_\vartheta$ . For  $\alpha = u_c(-\vartheta) = \alpha^\vee$ , the  $\lambda$ -inequality  $(c_-, \alpha) \geq j > 0$  is replaced with  $(c_-, \alpha) + 2 > j > 0$ ; respectively, the root  $[\alpha, (c_-, \alpha) + 1]$  is added to  $\lambda(\pi_c)$ . Here  $\alpha = \alpha^\vee \in R_-$  and  $(c_-, \alpha) = -(c, \vartheta) \geq 0$ .*

(iii) For any  $c \in P, r \in O'$ , one has  $\pi_r \pi_c = \pi_b$  where  $b = \pi_r((c))$ . Respectively,  $u_b = u_c u_r$ ,  $b = \omega_r + u_r^{-1}(c)$ ,  $b_- = c_- + u_c w_0(\omega_r)$ . In particular, the latter weight always belongs to  $P_-$ .

## 2. REDUCED DECOMPOSITIONS

We will discuss properties of the reduced decompositions in connection with the corresponding  $\lambda$ -sets and  $\lambda$ -sequences.

**2.1. Lambda-sequences.** Let us give an intrinsic description of the sequences  $\lambda(\hat{w})$ . See (1.32) for a more geometric approach based on the affine Weyl chambers.

**Main Theorem 2.1.** (i) A sequence  $\lambda = \{\tilde{\alpha}^l, \dots, \tilde{\alpha}^1\}$  of pairwise distinct roots from  $\tilde{R}_+$  can be represented in the form  $\lambda(\hat{w})$  for a certain  $\hat{w}$  if and only if the following six conditions are satisfied:

- (a)  $\{\tilde{\alpha} = \tilde{\alpha}^q + \tilde{\alpha}^r \notin [0, \mathbb{Z}], \tilde{\alpha} \in \tilde{R}_+\} \Rightarrow \tilde{\alpha} \in \lambda$ ;
- (b)  $\{\tilde{\alpha}^p = \tilde{\alpha}^q + \tilde{\alpha}^r\} \Rightarrow \{p \text{ is between } q, r \text{ in } \lambda\}$ ;
- (c)  $\{\tilde{\alpha}^p = \tilde{\beta} + \tilde{\gamma} \text{ for } \tilde{\beta}, \tilde{\gamma} \in \tilde{R}_+\} \Rightarrow \{\tilde{\beta} \in \lambda \text{ or } \tilde{\gamma} \in \lambda\}$ ;
- (d)  $\{\tilde{\alpha}^p = \tilde{\alpha}^q + \tilde{\gamma} \text{ for } \tilde{R}_+ \ni \tilde{\gamma} \notin \lambda\} \Rightarrow q < p$ ;
- (e)  $\{\tilde{\alpha}^p = [\alpha, \nu_\alpha j] \in \lambda, [\alpha, \nu_\alpha j'] \in \tilde{R}_+, 0 \leq j' < j\} \Rightarrow [\alpha, \nu_\alpha j'] \in \lambda$ ;
- (f)  $\{\tilde{\alpha}^p = [\alpha, \nu_\alpha j], \tilde{\alpha}^q = [\alpha, \nu_\alpha j'], 0 \leq j' < j\} \Rightarrow q < p$ .

(ii) Assuming that (a), (c), (e) hold for a set  $\lambda$ , called a  $\lambda$ -set, there exists at least one its ordering, called a  $\lambda$ -sequence, satisfying conditions (b), (d), (f). All such sequences are in one-to-one correspondence with reduced decompositions of  $\tilde{w} \in \tilde{W}$  such that  $\lambda(\tilde{w}) = \lambda$ .

(iii) Conditions (a, b, c, d) imply conditions

- (a')  $\{\tilde{\alpha}' = u\tilde{\alpha}^q + v\tilde{\alpha}^r \in \tilde{R}_+\} \Rightarrow \tilde{\alpha}' \in \lambda$ ;
- (b')  $\{\tilde{\alpha}^p = u\tilde{\alpha}^q + v\tilde{\alpha}^r\} \Rightarrow p \text{ is between } q, r$ ;
- (c')  $\{\tilde{\alpha}^p = u\tilde{\beta} + v\tilde{\gamma} \text{ for } \tilde{\beta}, \tilde{\gamma} \in \tilde{R}_+\} \Rightarrow \{\tilde{\beta} \in \lambda \text{ or } \tilde{\gamma} \in \lambda\}$ ;
- (d')  $\{\tilde{\alpha}^p = u\tilde{\alpha}^q + v\tilde{\gamma} \text{ for } \tilde{R}_+ \ni \tilde{\gamma} \notin \lambda\} \Rightarrow q < p$ ,

where  $u, v$  are positive rational numbers. These conditions coincide with (a, b, c, d) in the simply-laced case.

*Proof.* The induction in  $l$  will be used. Claims (i,ii) are obvious as  $l = 1$ . We will establish claims (i,ii) for  $l$  assuming that (i) holds for all  $1 \leq l' < l$ , that is the equivalence  $(a, b, c, d, e, f) \Leftrightarrow \{\lambda = \lambda(\widehat{w})\}$ , and that (ii) holds for such  $l'$ , that is the existence and description of the  $\lambda$ -sequences for any given  $\lambda$ -set,

Given  $\widehat{w}'$ , the product  $\widehat{w}'s_i$  is reduced if and only if  $\alpha_i \notin \lambda(\widehat{w}') = \{\tilde{\alpha}^{l-1}, \dots, \tilde{\alpha}^1\}$ . Then  $\lambda(\widehat{w}'s_i) = \{s_i(\lambda(\widehat{w}')), \alpha_i\}$ . Note that the last set is automatically positive (belongs to  $\widetilde{R}_+$ ) since  $s_i(\widetilde{R}_+) \cap \widetilde{R}_- = -\alpha_i$ . Indeed, the decomposition of any  $\widetilde{R}_+ \ni \tilde{\alpha} \neq \alpha^i$  in terms of simple roots in  $R_+$  and **imaginary roots**  $[0, \mathbb{Z}]$  contains either a simple root  $\alpha_j \neq \alpha_i$  with a positive coefficient or an imaginary root  $[0, m]$  with  $m > 0$ .

Let  $\lambda(\widehat{w})$  be the  $\lambda$ -sequences for a reduced decomposition  $\widehat{w} = \pi_r s_{i_l} \cdots s_{i_1}$ . Then the set  $\{s_{i_1}(\tilde{\alpha}^l), \dots, s_{i_1}(\tilde{\alpha}^2)\}$  is  $\lambda(\widehat{w}')$  for  $\widehat{w}' = \widehat{w}s_{i_1}$  of length  $l - 1$ , and we readily deduce  $(a - f)$  from the induction assumption. The only if statement for  $l$  is verified.

Let us check that  $\lambda = \{\tilde{\alpha}^p\}$  of length  $l$  satisfying  $(a, b, c, d, e, f)$  corresponds to a certain  $\widehat{w}$ . First of all,  $\tilde{\alpha}^1 \in \lambda$  must be a simple root. Otherwise, one can represent  $\tilde{\alpha}^p = \alpha_i + \tilde{\beta}$  for a certain simple  $\alpha_i$  where  $\tilde{\beta}$  is either from  $\widetilde{R}_+$  or is an imaginary root  $[0, \nu j]$  for some  $j > 0, \nu = \nu_{\tilde{\alpha}^p}$ . However, such  $\alpha_i$  must appear in the sequence  $\lambda$  before  $\tilde{\alpha}^1$ , which is impossible.

If only  $(a, c, e)$  are imposed for  $l' < l$ , then the same argument gives that  $\lambda$  considered as an unordered *set* contains at least one simple root.

Using the notations  $s^1 = s_{\tilde{\alpha}^1}$ ,  $\tilde{\alpha}' = s^1(\tilde{\alpha})$ , the roots  $(\tilde{\alpha}^p)' = s^1(\tilde{\alpha}^p)$  are all positive; see above. Let us establish that the sequence  $\lambda' = s^1(\{\tilde{\alpha}^l, \dots, \tilde{\alpha}^2\})$  satisfies  $(a, b, c, d, e, f)$ ; respectively, it satisfies  $(a, c, f)$  if  $\lambda$  and  $\lambda'$  are considered as unordered *sets*.

The claims  $(a, b)$  are obvious. Applying  $s^1$  to any  $[\beta, \nu_\beta j]$  diminishes  $j$  by a non-negative integer (by zero, if  $s^1$  is nonaffine), that gives  $(e, f)$ .

As for  $(c, d)$ , let  $(\tilde{\alpha}^p)' = \tilde{\beta}' + \tilde{\gamma}'$  for positive  $\tilde{\beta}'$  and  $\tilde{\gamma}'$ ; it suffices to consider only  $\tilde{\gamma}' = \tilde{\alpha}^1$ . Then

$$\tilde{\alpha}^p = \tilde{\beta} - \tilde{\alpha}^1 \Rightarrow \tilde{\beta} = \tilde{\alpha}^p + \tilde{\alpha}^1 \Rightarrow \{\tilde{\beta} = \tilde{\alpha}^q, q < p\} \Rightarrow \tilde{\beta}' \in \lambda',$$

where  $\tilde{\beta} = s^1(\tilde{\beta}')$ . It gives  $(c, d)$  if  $\tilde{\gamma}' = \tilde{\alpha}^1$ . Otherwise, both,  $\tilde{\beta}$  and  $\tilde{\gamma}$ , are positive, and  $(c, d)$  for  $\lambda'$  follow from those for  $\lambda$ .

Now we can use the induction assumption for  $\lambda'$ ; it gives that  $\lambda' = \lambda(\widehat{w}')$  and we can go from  $\widehat{w}'$  to  $\widehat{w} = \widehat{w}'s^1$  as above.

If only conditions  $(a, c, e)$  are imposed, then given *any* ordering of the set  $\lambda'$  of type  $(b, d, f)$ , the sequence  $\{s^1(\lambda'), \alpha^1\}$  satisfies  $(b, d, f)$  too. This is sufficient for establishing (ii).

The claims from (iii) are checked using the same induction consideration.  $\square$

Note that if  $\tilde{\gamma} = \sum_m \tilde{\alpha}^{q_m} \in \tilde{R}_+$  where  $\tilde{\alpha}^{q_m} \in \lambda(\hat{w})$ , then, obviously,  $\tilde{\gamma} \in \lambda(\hat{w})$ . It formally results from Theorem 2.1, (i) due to the known fact [B] that there exists a permutation of the indices  $m$ ,  $\tilde{\gamma} = \tilde{\alpha}^{q_1} + \tilde{\alpha}^{q_2} + \dots$ , such that all partial sums  $\tilde{\alpha}^{q_1}, \tilde{\alpha}^{q_1} + \tilde{\alpha}^{q_2}, \dots$  are from  $\lambda$ .

**2.2. Coxeter transformations.** We will prepare tools for studying transformations of the reduced decompositions. The elementary ones are the **Coxeter transformation** that are substitutions  $(\dots s_i s_j s_i) \mapsto (\dots s_j s_i s_j)$  in reduced decompositions of  $\hat{w} \in \hat{W}$  with the number of factors 2, 3, 4, 6 as  $\alpha_i$  and  $\alpha_j$  are connected by  $m_{ij} = 0, 1, 2, 3$  laces in the affine Dynkin diagram. They induce *reversing the order* in the corresponding segments (with 2, 3, 4, 6 roots) of the sequence  $\lambda(\hat{w})$ . The corresponding roots form a set identified with the set of positive roots of type  $A_1 \times A_1, A_2, B_2, G_2$  respectively. The action of Coxeter transformations in  $\lambda$ -sets plays an important role in the paper.

**Proposition 2.2.** (i) *Given  $\hat{w} \in \hat{W}$ , the roots  $\tilde{\alpha}$  that may appear in the beginning, the first roots, of the sequence  $\lambda(\hat{w}) = \{\dots, \tilde{\alpha}\}$  for different reduced decompositions of  $\hat{w}$  are exactly simple roots  $\tilde{\alpha} = \alpha_i \in \lambda(\hat{w})$ . The last roots, i.e.,  $\tilde{\alpha}$  such that  $\lambda(\hat{w}) = \{\tilde{\alpha}, \dots\}$  for a suitable reduced decomposition are exactly  $\tilde{\alpha} = -\hat{w}^{-1}(\alpha_i)$  for  $\alpha_i \in \lambda(\hat{w}^{-1}) = -\hat{w}(\lambda(\hat{w}))$ .*

(ii) *The last roots of  $\lambda(\hat{w})$  are also exactly  $\tilde{\alpha} \in \lambda(\hat{w})$  satisfying the following two conditions:*

- (a)  $\tilde{\alpha} \neq \tilde{\beta} + \tilde{\gamma}$  for any two roots  $\tilde{\beta}, \tilde{\gamma} \in \lambda(\hat{w})$ ,
- (b)  $\tilde{\gamma} \neq \tilde{\alpha} + \tilde{\beta}$  for any  $\tilde{\gamma} \in \lambda(\hat{w})$ ,  $\tilde{R}_+ \cup [0, \mathbb{N}] \ni \tilde{\beta} \notin \lambda(\hat{w})$ .

(iii) *Given a reduced decomposition of  $\hat{w}$ , if  $\tilde{\gamma} = u\tilde{\alpha} + v\tilde{\beta}$  for  $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} \in \lambda(\hat{w})$ ,  $\mathbb{Q} \ni u, v > 0$ , and  $\tilde{\alpha}, \tilde{\gamma}$  are neighboring in  $\lambda(\hat{w})$ , then proper Coxeter transformations in the segment from  $\tilde{\beta}$  to  $\tilde{\gamma}$  ( $\tilde{\gamma}$  is excluded) make  $\tilde{\beta}$  next to  $\tilde{\gamma}$ , i.e., make the triple  $\tilde{\beta}, \tilde{\gamma}, \tilde{\alpha}$  a connected segment in the resulting  $\lambda$ -sequence.*

*Proof.* Claim (i) is simple and well known. The demonstration of (ii) is by induction; we assume that  $\hat{w} = \hat{w}' s_i$  as  $l(\hat{w}) = l(\hat{w}') + 1$  and apply (ii) for  $\hat{w}'$ . It is as follows.



Since  $s_i(\tilde{\alpha}') > 0$  for all  $\tilde{\alpha}' \in \tilde{R}_+ \setminus \{\alpha_i\}$ , a root  $\tilde{\alpha} \in \lambda(\hat{w})$  satisfying  $(a, b)$  for  $\hat{w}$  equals  $s_i(\tilde{\alpha}')$  for  $\tilde{\alpha}' \in \lambda(\hat{w}')$  satisfying  $(a, b)$  for  $\hat{w}'$  unless  $\tilde{\alpha} = \alpha_i$  ( $\tilde{\beta} \neq \alpha_i$  because  $\alpha_i \in \lambda(\hat{w})$ ). Thus  $\tilde{\alpha} \neq \alpha_i$  is the *last* for a certain reduced decomposition of  $\hat{w}'$  multiplied by  $s_i$  on the right.

We need only to check that  $\alpha_i$  can be made the *last* in a sequence  $\lambda(\hat{w})$  if

$$(2.40) \quad \tilde{\gamma} \neq \alpha_i + \tilde{\beta} \text{ for any } \tilde{\gamma} \in \lambda(\hat{w}) \not\equiv \tilde{\beta} \in \tilde{R}_+ \cup [0, \mathbb{N}].$$

By induction, the last two roots of  $\lambda(\hat{w})$  can be made  $\tilde{\delta}, \alpha_i, \dots$ , ( $\lambda(\hat{w}) = \{\tilde{\alpha}, \alpha_i, \dots\}$ ) for a proper reduced decomposition of  $\hat{w}$ . Here  $\tilde{\delta}$  is the end of  $\lambda(\hat{w})$ ; it satisfies  $s_i(\tilde{\delta}) > 0$  since  $\tilde{\delta} = s_i(\tilde{\delta}')$  for  $\tilde{\delta}' \stackrel{\text{def}}{=} s_i(\tilde{\delta}) \in \lambda(\hat{w}') \subset \tilde{R}_+$ .

Setting  $\tilde{\delta}' = s_i(\tilde{\delta}) = \tilde{\delta} + m\alpha_i$ , either  $m = 0$  and we can simply transpose  $\tilde{\delta}$  and  $\alpha_i$  in  $\lambda(\hat{w})$ , or  $m > 0$ , or  $m < 0$ . In the case  $m > 0$ ,  $\tilde{\delta}' \in \lambda(\hat{w})$  and it must be between  $\tilde{\delta}$  and  $\alpha_i$ , which is impossible (they are neighbors). If  $m < 0$  and  $\tilde{\delta}' \in \lambda(\hat{w})$ , then  $\tilde{\delta}$  must be between  $\tilde{\delta}'$  and  $\alpha_i$ , which is impossible too. Thus  $m = -p < 0$  and  $\tilde{\delta}' \notin \lambda(\hat{w})$ .

Finally,  $\alpha_i + \tilde{\beta} = \tilde{\delta}$  where the root  $\tilde{\beta} \stackrel{\text{def}}{=} \tilde{\delta}' + (p-1)\alpha_i$  belongs to  $\tilde{R}_+ \cup [0, \mathbb{N}]$  (see [B]). If here  $\tilde{\beta} \in \lambda(\hat{w})$ , then  $\tilde{\delta}$  must be between  $\tilde{\beta}$  and  $\alpha_i$ , which is impossible because  $\tilde{\delta}$  was the end of the sequence  $\lambda(\hat{w})$ . If  $\tilde{\beta} \notin \lambda(\hat{w})$ , then it contradicts assumption (2.40); (ii) is checked.

Now let  $u\tilde{\alpha} + v\tilde{\beta} = \tilde{\gamma}$  for the roots in  $\lambda(\hat{w})$  as  $u, v > 0$ . We are going to make them consecutive in a proper reduced decomposition. Note that  $u, v > 0$  is necessary and sufficient to make  $\tilde{\gamma}$  between  $\tilde{\alpha}$  and  $\tilde{\beta}$ .

One can suppose that  $\tilde{\alpha} = \alpha_i$  is the *first* in  $\lambda(\hat{w})$ . Then  $\tilde{\gamma}$  is the *second* in this sequence and the reduced decomposition reads as  $\hat{w} = \dots s_j s_i$  for  $j$  such that  $\tilde{\gamma} = s_i(\alpha_j) = m\alpha_i + \alpha_j$  for  $m \in \mathbb{N}$  and  $\alpha_i, \alpha_j$  are connected in the affine Dynkin diagram. Continuing, we can find a reduced decomposition of  $\hat{w}$  in the form  $\hat{w} = \tilde{u}\tilde{v}$ ,  $\tilde{v} = \dots s_i s_j s_i$ ,  $l(\tilde{u}\tilde{v}) = l(\tilde{u}) + l(\tilde{v})$  where  $\tilde{v}$  is the longest possible product of  $s_i$  and  $s_j$ , equivalently,  $\alpha_i, \alpha_j \notin \lambda(\tilde{u})$ .

Let us assume that  $\tilde{\beta} \in \tilde{v}^{-1}(\lambda(\tilde{u}))$ . Since  $\tilde{v}(\tilde{\beta})$  is a (positive) linear combination of  $\alpha_i, \alpha_j$ , then either  $\alpha_i$  or  $\alpha_j$  must belong to  $\lambda(\tilde{u})$ , a contradiction. Hence  $\tilde{\beta} \in \lambda(\tilde{v})$ , which proves (iii).  $\square$

**Comment.** Claim (ii) is a demonstration that Theorem 2.1 is sufficient for a *complete* characterization of the *last roots* of the  $\lambda$ -sequences.

Note that one can use here the plane geometric interpretation of the reduced decompositions from [C2] for affine classical root systems (see there the reference concerning the non-affine case and the so-called reflection equation).  $\square$

For  $\lambda$  satisfying Theorem 2.1, one can introduce **quasi-simple roots**  $\tilde{\beta} \in \lambda$  that are not sums of the roots from  $\lambda$  as in (a). Arbitrary  $\tilde{\alpha} \in \lambda$  are their sums, but, generally, they are not linearly independent vectors; for example, there are four quasi-simple roots in  $\lambda(w)$  for  $w = (4231) \in \mathbf{S}_4$  (the case of  $A_3$ ).

It is possible to express the set  $\{\tilde{\beta}\}$  of quasi-simple roots of  $\tilde{w}$  in terms of  $\{\tilde{\beta}'\}$  for  $\tilde{w}'$  as  $\tilde{w} = \tilde{w}'s_i$ ,  $l(\tilde{w}) = l(\tilde{w}') + 1$ :

$$(2.41) \quad \{\tilde{\beta}\} = \{\alpha_i\} \cup \{s_i(\tilde{\beta}') \mid \tilde{\beta}' + \alpha_i \notin \lambda(\tilde{w}')\}.$$

The following variant of this definition (they coincide in the simply-laced case) has applications to the Bruhat ordering. We call  $\tilde{\beta} \in \lambda$  a **pseudo-simple root** if  $m\tilde{\beta}$  is not a sum of roots in  $\lambda$  for any  $m \in \mathbb{N}$ .

**Proposition 2.3.** (i) Given a  $\lambda$ -sequence  $\lambda = \lambda(\hat{w})$ , the indices  $\{p\}$  of pseudo-simple roots  $\tilde{\alpha}^p$  (see (1.21)) are exactly those satisfying the condition  $l(\hat{w}') = l - 1$  for  $\hat{w}'$  obtained from  $\hat{w}$  by deleting  $s_{i_p}$  in  $\hat{w} = \pi_r s_{i_l} \cdots s_{i_1}$ :

$$\hat{w}' = \pi_r s_{i_l} \cdots s_{i_{p+1}} s_{i_{p-1}} \cdots s_{i_1} = \hat{w} s^p \quad \text{for } s^p = s_{\tilde{\alpha}^p}$$

(ii) The set  $\{\tilde{\beta}\}$  of pseudo-simple roots of  $\tilde{w} = \tilde{w}'s_i$  such that  $l(\tilde{w}) = l(\tilde{w}') + 1$  is as follows. For the set of pseudo-simple roots  $\{\tilde{\beta}'\}$  for  $\tilde{w}'$ ,

$$\{\tilde{\beta}\} = \{\alpha_i\} \cup \{s_i(\tilde{\beta}') \mid m' \tilde{\beta}' + \alpha_i \notin \lambda(\tilde{w}') \text{ for any } m' \in \mathbb{N}\}.$$

(iii) Arbitrary roots from  $\lambda(\hat{w})$  are linear combinations of pseudo-simple roots with positive rational coefficients.

*Proof.* The condition  $l(\hat{w}') = l - 1$  is equivalent to the positivity  $s^p(\tilde{\alpha}^q) \in \tilde{R}_+$  for  $q = p + 1, \dots, l$ ; see (1.16). If  $s^p(\tilde{\alpha}^q) < 0$  then  $-\tilde{\beta} = \tilde{\alpha}^q - m\tilde{\alpha}^p < 0$  for  $m = ((\alpha^p)^\vee, \alpha^q) > 0, \tilde{\beta} > 0$  and  $m\tilde{\alpha}^p = \tilde{\beta} + \tilde{\alpha}^q$ . However,  $-\tilde{\beta}$  cannot appear in  $\lambda$ -sets and has to coincide with  $-\tilde{\alpha}^r$  for some  $r < p$ ; see (1.17). Thus  $m\tilde{\alpha}^p = \tilde{\alpha}^r + \tilde{\alpha}^q$  and  $\tilde{\alpha}^p$  is not pseudo-simple in  $\lambda$ .

Let us begin now with  $\tilde{\alpha}^p$  such that  $m\tilde{\alpha}^p = \tilde{\alpha}^r + \tilde{\alpha}^q$  for  $q > p > r$ . Considering a root subsystem of rank two containing  $\tilde{\alpha}^p, \tilde{\alpha}^r, \tilde{\alpha}^q$  as

positive roots, the coefficient  $m$  can be 1 for  $A_2$ , 1, 2 for  $B_2$ , 1, 2, 3 for  $G_2$ . In either case, the root

$$s^p(\tilde{\alpha}^q) = \tilde{\alpha}^q - m' \tilde{\alpha}^p = (m - m') \tilde{\alpha}^p - \tilde{\alpha}^r \quad \text{for } m' = ((\alpha^p)^\vee, \alpha^q)$$

is negative.

Claim (ii) is parallel to (2.41). Claim (iii) follows from (ii). Indeed,  $\alpha_i$  belongs to the set of pseudo-simple roots in  $\lambda(\hat{w})$ . Also, given a pseudo-simple root  $\tilde{\beta}' \in \lambda(\hat{w}')$ , the root  $s_i(\tilde{\gamma}')$  will be pseudo-simple in  $\lambda(\hat{w})$  for the last  $\tilde{\gamma}'$  in the sequence  $\lambda(\hat{w}')$  satisfying  $\tilde{\gamma}' \in u\tilde{\beta}' + v\alpha_i$  as  $u, v \in \mathbb{Q}, u > 0$ .  $\square$

**2.3. Theorem about triples.** We are going to discuss a connection of the non-*quasi-simple* roots and the Coxeter transformations; the latter become Coxeter *permutations* in the context of  $\lambda$ -sets. The theorem below is expected to be an important tool for the classification of semisimple representations of DAHA and similar questions. It clearly demonstrates why dealing with the intertwining operators for arbitrary root systems is significantly more difficult than in the  $A_n$ -theory (where much is known). The classical theory of root systems [B, Hu] is uniform at level of the generators and relations. However, if the “relations of relations” are considered the root systems behave differently; the simplest are  $A_n$  and the rank two systems.

First of all, positive roots  $\alpha, \beta$  from a rank two root system  $R^2$  of type  $A_2, B_2, G_2$  are simple if and only if

$$(2.42) \quad \tilde{\alpha} + \tilde{\beta} \in R^2 \text{ and } |\tilde{\alpha}| \neq |\tilde{\beta}| \text{ for } B_2, G_2, \text{ where } |\tilde{\alpha}|^2 = 2\nu_\alpha.$$

Note that  $\tilde{\alpha} + \tilde{\beta}$  is always a short root for such  $\tilde{\alpha}, \tilde{\beta}$ .

Given a reduced decomposition of  $\hat{w}$ , the endpoints in  $\lambda(\hat{w})$  corresponding to (complete) Coxeter sub-products  $(\cdots s_i s_j s_i)$  are such  $\tilde{\alpha}, \tilde{\beta}$ . Vice versa, if  $\tilde{\gamma} = \tilde{\alpha} + \tilde{\beta} \in \tilde{R}_+$  and also  $|\tilde{\gamma}| = |\tilde{\alpha}| = |\tilde{\beta}|$  as  $|\tilde{\alpha}| = |\tilde{\beta}|$ , then  $\tilde{\alpha}, \tilde{\beta}$  come from (2.42) for some  $R^2$  unless  $\tilde{R}$  is of type  $\tilde{G}_2$ .

If  $\tilde{R}$  is the affine system of type  $\tilde{G}_2$  and  $|\tilde{\gamma}| = |\tilde{\alpha}| = |\tilde{\beta}|$ , then we need to assume additionally that  $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$  are all short and there are no long roots among their linear combinations, i.e., that they do not belong to any (finite) subsystem of  $\tilde{R}$  of type  $G_2$ . We call such triples  $A_2$ -*pure-short*.

In the simply-laced case, we set (technically)  $lng = sht$ , i.e., any conditions that certain roots are long or short are disregarded as  $\tilde{R} = \tilde{A}, \tilde{D}, \tilde{E}$  in the following theorem and below. Recall that we use the notation  $\tilde{R} = \tilde{A}_n, \tilde{B}_n, \dots, \tilde{G}_2$  for  $\tilde{R}$  corresponding to  $R = A_n, B_n, C_n$ .

A connected part of the sequence  $\lambda(\hat{w})$  isomorphic to the sequence of all positive roots of type  $A_2, B_2$  or  $G_2$  will be called a **segment of rank two** in the next theorem. Concerning *root subsystems* used in this theorem, they can be arbitrary; it suffices to suppose that they are intersections of  $\tilde{R}$  with  $\mathbb{Q}$ -subspaces in  $\mathbb{Q}[\tilde{R}]$ .

**Main Theorem 2.4.** *Given a reduced decomposition of  $\hat{w} \in \widehat{W}$ , let us assume that  $\tilde{\alpha} + \tilde{\beta} = \tilde{\gamma}$  for the roots  $\dots, \tilde{\beta}, \dots, \tilde{\gamma}, \dots, \tilde{\alpha} \dots$  in  $\lambda(\hat{w})$  ( $\tilde{\alpha}$  appears the first), where only the following combinations of their lengths are allowed:*

$$(2.43) \quad (a) \text{ } lng + sht = sht \text{ or } sht + lng = sht \text{ (all systems } \tilde{R}),$$

$$(2.44) \quad (b) \text{ } lng + lng = lng \text{ (}\tilde{B}, \tilde{F}_4\text{) or } sht + sht = sht \text{ (}\tilde{C}, \tilde{F}_4\text{),}$$

$$(2.45) \quad (c) \text{ the roots } \tilde{\alpha}, \tilde{\beta} \text{ are } A_2\text{-pure-short when } \tilde{R} = \tilde{G}_2.$$

(i) Let  $[\tilde{\beta}, \tilde{\alpha}]$  be a segment in  $\lambda(\hat{w})$  from  $\tilde{\alpha}$  to  $\tilde{\beta}$ . There exists  $\tilde{u} \in \widehat{W}$  such that  $\tilde{u}(\tilde{\gamma}) = \tilde{\gamma}$  and the Coxeter transforms in  $[\tilde{u}(\tilde{\beta}), \tilde{u}(\tilde{\alpha})]$  can be used to make the triple  $\tilde{u}(\tilde{\alpha}), \tilde{u}(\tilde{\beta}), \tilde{u}(\tilde{\gamma}) = \tilde{\gamma}$  part of a segment  $L$  of rank 2 in  $\lambda(\hat{w})$ . Moreover, one can assume here that  $\tilde{u} = s_{j_l} \cdots s_{j_1}$  and all consecutive products  $\tilde{u}_m = s_{j_m} \cdots s_{j_1}$  ( $\tilde{u}_0 = id$ ) leave  $\tilde{\gamma}$  invariant for  $m = 1, \dots, l$  and

$$\tilde{u}_m(\tilde{\alpha}) \in \lambda(\hat{w}) \ni \tilde{u}_m(\tilde{\beta}), [\tilde{u}_m(\tilde{\alpha}), \tilde{u}_m(\tilde{\beta})] \subset [\tilde{u}_{m-1}(\tilde{\alpha}), \tilde{u}_{m-1}(\tilde{\beta})],$$

where the Coxeter transforms can be used in  $[\tilde{u}_{m-1}(\tilde{\alpha}), \tilde{u}_{m-1}(\tilde{\beta})]$  before finding the next  $[\tilde{u}_m(\tilde{\alpha}), \tilde{u}_m(\tilde{\beta})]$ .

One can take  $\tilde{u} = id$  in cases (a) for  $\tilde{R} = \tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \tilde{F}_4, \tilde{G}_2$  and also for the system  $\tilde{G}_2$  under (c).

(ii) For the triples of type (b), such  $L$  does not exist with  $\tilde{u} = id$  if (and only if unless for  $\tilde{C}_{n \geq 4}$ ) a root subsystem  $R^3 \subset \tilde{R}$  of type  $B_3$  or  $C_3$  ( $m = 1, 2$ ) can be found such that

$$(2.46) \quad \tilde{\beta} = \epsilon_1 + \epsilon_3, \tilde{\alpha} = \epsilon_2 - \epsilon_3, \epsilon_1 - \epsilon_2 \notin [\tilde{\beta}, \tilde{\alpha}] \not\equiv m\epsilon_3, \text{ or}$$

$$(2.47) \quad \tilde{\beta} = \epsilon_2 + \epsilon_3, \tilde{\alpha} = \epsilon_1 - \epsilon_3, \epsilon_1 - \epsilon_2 \notin [\tilde{\beta}, \tilde{\alpha}] \not\equiv m\epsilon_3;$$

the notation is from [B], the positivity in  $R^3$  is induced from  $\tilde{R}_+$ . Equivalently, the sequence  $[\tilde{\beta}, \tilde{\alpha}] \cap R_+^3$  (with the natural ordering) must be

$$(2.48) \quad \{ \epsilon_1 + \epsilon_3, m\epsilon_1, \epsilon_2 + \epsilon_3, \tilde{\gamma} = \epsilon_1 + \epsilon_2, \epsilon_1 - \epsilon_3, m\epsilon_2, \epsilon_2 - \epsilon_3 \} \text{ or}$$

$$(2.49) \quad \{ \epsilon_2 + \epsilon_3, m\epsilon_2, \epsilon_1 + \epsilon_3, \tilde{\gamma} = \epsilon_1 + \epsilon_2, \epsilon_2 - \epsilon_3, m\epsilon_1, \epsilon_1 - \epsilon_3 \}$$

up to Coxeter transforms in  $R^3$  and changing the order of all roots in (2.48) to the opposite. If  $\tilde{\alpha}$  is assumed simple in  $R^3$ , then only (2.46), (2.48) may occur.

If such  $R^3$  exist, one can still take  $\tilde{u} = id$  using Coxeter transforms in the whole  $\lambda(\hat{w})$  if the latter set contains either  $\epsilon_1 - \epsilon_2$  or  $m\epsilon_3$  for every such  $R^3$ .

(iii) For the root system  $\tilde{R}$  of type  $\tilde{C}_{n \geq 4}, \tilde{D}_{n \geq 4}$  or  $\tilde{E}_{6,7,8}$ , let us assume that  $\tilde{\alpha}$  is simple in  $\tilde{R}_+$ . Then such  $L$  does not exist with  $\tilde{u} = id$  if (and only if unless for  $\tilde{C}$ ) a root subsystem  $R^4 \subset \tilde{R}$  of type  $D_4$  can be found such that

$$(2.50) \quad \begin{aligned} \tilde{\beta} &= \epsilon_1 + \epsilon_3, \tilde{\gamma} = \epsilon_1 + \epsilon_2, \tilde{\alpha} = \epsilon_2 - \epsilon_3, \\ \{ \epsilon_1 - \epsilon_2, \epsilon_3 - \epsilon_4, \epsilon_3 + \epsilon_4 \} \cap [\tilde{\beta}, \tilde{\alpha}] &= \emptyset; \end{aligned}$$

the notation is from [B], the positivity in  $R^4$  is induced from  $\tilde{R}_+$ . Equivalently, the sequence  $[\tilde{\beta}, \tilde{\alpha}] \cap R_+^4$  must be

$$(2.51) \quad \begin{aligned} \{ \tilde{\beta} = \epsilon_1 + \epsilon_3, \epsilon_1 - \epsilon_4, \epsilon_1 + \epsilon_4, \epsilon_1 - \epsilon_3, \tilde{\gamma} = \epsilon_1 + \epsilon_2, \\ \epsilon_2 + \epsilon_3, \epsilon_2 + \epsilon_4, \epsilon_2 - \epsilon_4, \tilde{\alpha} = \epsilon_2 - \epsilon_3 \} \end{aligned}$$

up to Coxeter transforms in  $R^4$ . Equivalently,  $[\tilde{\beta}, \tilde{\alpha}] \cap R_+^4$  is the  $\lambda$ -set of  $s_{\theta^4}$  in  $R_+^4$  for the maximal root  $\theta^4$ . Equivalently,  $\tilde{\alpha} = \alpha_2^4$ ,  $\tilde{\gamma} = \theta^4$ ,  $\tilde{\beta} = \theta^4 - \alpha_2^4$  and also  $\alpha_2^4$  (a unique simple root non-orthogonal to  $\theta^4$ ) is the only simple root from  $R^4$  in  $[\tilde{\beta}, \tilde{\alpha}]$ .

In case of  $\tilde{C}_{n \geq 4}$  (when  $\tilde{\Gamma}$  contains a subdiagram of type  $D_4$ ), either (2.50)-(2.51) must hold or those from (ii) for  $R^3$  if such  $L$  does not exist with  $\tilde{u} = id$ .

If  $\tilde{\alpha}$  is not assumed simple in  $\tilde{R}_+$ , then the condition is that  $\tilde{\gamma} = \theta^4$  and  $[\tilde{\beta}, \tilde{\alpha}] \cap R_+^4$  contains a unique simple root from  $R_+^4$ .

**2.4. Discussion.** Before proving the theorem, let us discuss some corollaries and general facts that we will need to clarify and verify claims (i,ii,iii).

We do not give the complete list of possibilities for (iii) without the assumption that  $\tilde{\alpha}$  is simple. It is analogous to that in (ii). Actually, it suffices to assume that it is simple in  $R_+^4$  because one can always switch to considering  $(\hat{w}')^{-1}[\tilde{\beta}, \tilde{\alpha}]$  where  $\hat{w}'$  is the portion of the reduced decomposition of  $\hat{w}$  before  $\tilde{\alpha}$ . However, sometimes it is convenient to avoid making  $\tilde{\alpha}$  simple; the next comment can be readily extended to (iii).

**Comment.** Note that (2.48) is the only case that may occur if  $\tilde{\alpha}$  is assumed to be simple in  $R_+^3$ . In this case, if  $\epsilon_1 - \epsilon_2$  or  $m\epsilon_3$  appear in  $\lambda(\hat{w})$ , then it can happen only *after* the last root in this sequence,  $\epsilon_1 + \epsilon_3$ . For instance,  $m\epsilon_3$  must be after  $m\epsilon_1$  since  $\epsilon_3 + (\epsilon_1 - \epsilon_3) = \epsilon_1$  implies that it is after  $\epsilon_1 + \epsilon_3$ . As for (2.49),  $m\epsilon_3$  (if present in  $\lambda(\hat{w})$ ) appears after the last root,  $\epsilon_2 + \epsilon_3$ . Also,  $\lambda(\hat{w})$  *must* contain  $\epsilon_1 - \epsilon_2$  before  $\epsilon_1 - \epsilon_3$ , since  $\epsilon_1 - \epsilon_3 = (\epsilon_1 - \epsilon_2) + (\epsilon_2 - \epsilon_3)$  and  $\epsilon_2 - \epsilon_3$  is after  $\epsilon_1 - \epsilon_3$  in (2.49).

Similarly, if the inverse of (2.48),

$$(2.52) \quad \{ \epsilon_2 - \epsilon_3, m\epsilon_2, \epsilon_1 - \epsilon_3, \tilde{\gamma} = \epsilon_1 + \epsilon_2, \epsilon_2 + \epsilon_3, m\epsilon_1, \epsilon_1 + \epsilon_3 \},$$

belongs to  $\lambda(\hat{w})$ , then the latter sequence *must* contain  $\epsilon_1 - \epsilon_2$  and  $m\epsilon_3$  (before  $\epsilon_1 + \epsilon_3$ ). For instance,  $(\epsilon_1 - \epsilon_2) + (\epsilon_2 + \epsilon_3) = \epsilon_1 + \epsilon_3$  results in  $\epsilon_1 - \epsilon_2 \in \lambda(\hat{w})$ .  $\square$

**Example of  $E_6$ .** The following example seems a good illustration of (iii). Let  $w = s_\theta$  for the maximal root  $\theta = \omega_2$  in  $R$  of type  $E_6$ . Then  $\alpha_2$  cannot be moved in the triple  $\{\beta = \theta - \alpha_2, \gamma = \theta, \alpha = \alpha_2\}$  from its first position and these roots cannot be made neighboring in  $[\beta, \alpha] \in \lambda(s_\theta)$ . Thus there must exist a 9-root set of type  $D_4$  from (2.51). In notation from [B], the simple roots from  $R^4$  are as follows:  $\epsilon_1 - \epsilon_2 = \alpha_4$ ,  $\epsilon_2 - \epsilon_3 = \alpha_2$  and  $\epsilon_3 - \epsilon_4 = \alpha_3 + \alpha_4 + \alpha_5$ ,  $\epsilon_3 + \epsilon_4 = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$ . The complete description is as follows (we show the roots from  $E_6$  under the corresponding ones from the 9-set):

$\epsilon_1 + \epsilon_3,$	$\epsilon_1 - \epsilon_4,$	$\epsilon_1 + \epsilon_4,$	$\epsilon_1 - \epsilon_3,$	$\epsilon_1 + \epsilon_2,$	$\epsilon_2 + \epsilon_3,$	$\epsilon_2 + \epsilon_4,$	$\epsilon_2 - \epsilon_4,$	$\epsilon_2 - \epsilon_3 :$
12321,	11211,	01210,	12221,	12321,	00100,	11111,	01110,	00000.
1	1	1	1	2	1	1	1	1

$\square$

**Example of  $F_4$ .** Let  $w = s_\vartheta$  for the *maximal short root*  $\vartheta = \omega_1$  in  $R$  of type  $F_4$ . Then  $\alpha_1$  cannot be moved from its first position in the triple  $\{\beta = \vartheta - \alpha_1, \gamma = \vartheta, \alpha = \alpha_1\}$ . According to the theorem, there must exist a 7-root set of type  $C_3$  from (2.48). In notation from [B], it is as follows: .

$$\begin{array}{ccccccc} \epsilon_1 + \epsilon_3, & 2\epsilon_1, & \epsilon_2 + \epsilon_3, & \epsilon_1 + \epsilon_2, & \epsilon_1 - \epsilon_3, & 2\epsilon_2, & \epsilon_2 - \epsilon_3 : \\ 1231, & 2342, & 0121, & 1232, & 1110, & 0122, & 0001. \end{array}$$

□

Note, that  $\{\epsilon_i\}$  are the roots from Theorem 2.4 in these two examples, i.e., they are the basic vectors used in [B] to describe  $R^4$  and  $R^3$  correspondingly (not  $\{\epsilon_i\}$  for  $E_6$  and  $F_4$ ).

One can readily find all *reflections*  $s_{\tilde{\gamma}}$  with the endpoints  $\alpha = \alpha_i, \tilde{\beta} = \tilde{\gamma} - \alpha_i \in \lambda(s_{\tilde{\gamma}})$  that are *non-movable* inside  $\lambda(s_{\tilde{\gamma}})$  under the Coxeter transformations. The examples above can be generalized as follows.

**Proposition 2.5.** (i) Let us assume that the  $\lambda$ -sequence  $\lambda(s_{\tilde{\gamma}})$  for  $\tilde{\gamma} \in \tilde{R}_+$  has a unique beginning  $\tilde{\gamma} = \alpha_i$  for  $0 \leq i \leq n$ . Then  $\tilde{\beta} = -s_{\tilde{\gamma}}(\alpha_i)$  is its unique end. Provided that the lengths of  $\tilde{\gamma}$  and  $\alpha_i$  coincide and the nonaffine components of these roots are not proportional,  $\tilde{\beta} = \tilde{\gamma} - \alpha_i$  and  $\{\tilde{\beta}, \tilde{\gamma}, \tilde{\alpha}\}$  form an  $A_2$ -triple.

(ii) Let  $\tilde{\gamma} = [\gamma, \nu_\gamma j]$  for  $\gamma \in R_+, j \geq 0$ . Then the conditions from (i) are satisfied only for  $i > 0$  and if and only if

- a)  $(\gamma, \alpha_i) > 0$  for a unique  $1 \leq i \leq n$  ( $\leq 0$  otherwise),
- b) moreover,  $|\gamma| = |\alpha_i|$  and  $\gamma \neq \alpha_i$  for such  $\alpha_i$ .

Here  $j \geq 0$  can be arbitrary.

(iii) Let  $\tilde{\gamma} = [-\gamma, \nu_\gamma j]$  for  $\gamma \in R_+, j > 0$ . Then (i) holds only for  $i > 0$  and if and only if  $\gamma$  is either a maximal short root or a maximal (long) root,  $\vartheta'$  or  $\theta'$ , for a root subsystem  $R'$  corresponding to a connected subdiagram  $\Gamma' \in \Gamma$  such that

- a)  $\alpha_{\Gamma'} \notin \Gamma'$  for  $\alpha_{\Gamma'} \in \Gamma$  linked to  $\alpha_0$  in the affine Dynkin diagram  $\tilde{\Gamma}$ ,
- b) there exists a unique  $\Gamma \ni \alpha_m \notin \Gamma'$  connected with  $\Gamma'$  by a link.

Here  $i = m$  and  $\vartheta'$  or  $\theta'$  are chosen to ensure that  $|\gamma| = |\alpha_m|$  ( $\Gamma'$  must contain roots of length  $|\alpha_m|$ );  $j > 0$  can be arbitrary.

*Proof.* Claim (ii) is straightforward. The uniqueness of  $\alpha_i$  gives that  $\alpha_0 \notin \lambda(s_{\tilde{\gamma}})$ , i.e.,  $i > 0$ . Indeed,  $s_{\tilde{\gamma}}(\alpha_0) = \alpha_0 + 2 \frac{(\gamma, \vartheta)}{(\gamma, \gamma)} \gamma$ , and the latter

is a positive root since  $(\gamma, \vartheta) \geq 0$ . However, at least one simple root must sit in  $\lambda(s_{\tilde{\gamma}})$ ; therefore, it can be only nonaffine.

Similarly,  $(\gamma, \alpha_{i'}) \leq 0$  for  $i' = 1, \dots, n$  unless  $i' = i$ , (when this inner product must be strictly positive). Otherwise,  $\alpha_{i'} \in \lambda(s_{\tilde{\gamma}})$ , which contradicts the uniqueness. The condition  $|\gamma| = |\alpha_i|$  here is necessary and sufficient to ensure that we really deal with an  $A_2$ -triple.

Assuming now that  $\tilde{\gamma} = [-\gamma, \nu_{\gamma}j]$  for  $\gamma > 0, j > 0$ , let us check that  $i > 0$  in (i). If  $i = 0$  then (i) holds if and only if  $\gamma$  is short,  $(\gamma, \vartheta) = 1$  and  $(\gamma, \alpha_{i'}) \geq 0$  for all  $i' = 1, \dots, n$ . However, the latter implies that  $\gamma = \vartheta$ , which contradicts to  $(\gamma, \vartheta) = 1$ . Thus  $(\gamma, \vartheta) = 0$ , which holds if and only if the decomposition of  $\gamma$  does not contain  $\alpha_{0'}$  connected with  $\alpha_0$  in  $\tilde{\Gamma}$  by a link.

Let  $\Gamma' \subset \Gamma$  be the *support* of  $\gamma$  (the set of all simple roots appearing in its decomposition). Then any simple root  $\alpha_m$  from (b) can be the *beginning* of the sequence  $\lambda(s_{\tilde{\gamma}})$  due to  $(\gamma, \alpha_m) < 0$ , which implies that

$$s_{\tilde{\gamma}}(\alpha_m) = \alpha_m + 2 \frac{(\alpha_m, \gamma)}{(\gamma, \gamma)} \tilde{\gamma} \in \tilde{R}_- \quad \text{and} \quad \alpha_m \in \lambda(s_{\tilde{\gamma}}).$$

At least one such  $\alpha_m$  exists. Thanks to the uniqueness, one such  $\alpha_m$  can exist. Moreover,  $(\gamma, \alpha_{i'}) \geq 0$  must hold for all  $\alpha_{i'} \in \Gamma'$ ; otherwise, there will be other simple roots in  $\lambda(s_{\tilde{\gamma}})$ . These conditions imply that either  $\gamma = \vartheta'$  as  $\alpha_m$  is short or  $\gamma = \theta'$  (the maximal positive root in  $\Gamma'$ ) as  $\alpha_m$  is long, which results from  $|\gamma| = |\alpha_m|$ . Here we use that  $\vartheta$  is a minimal positive root in  $P_+$ .  $\square$

We can now describe all non-affine roots  $\tilde{\gamma} \in \tilde{R}_+$  such that  $\tilde{\gamma}$  and the ends of  $\lambda(s_{\tilde{\gamma}})$  form a *non-movable*  $A_2$ -triples; the examples of  $E_6$  and  $F_4$  considered above correspond to  $\tilde{\gamma} = \vartheta$ . Proposition 2.5, (iii) gives a complete description of such roots in the form  $\tilde{\gamma} = [-\gamma, \nu_{\gamma}j] > 0$  for  $\gamma > 0$ . Therefore we can restrict ourselves with  $\tilde{\gamma} = [\gamma, \nu_{\gamma}j] > 0$ . Moreover, it suffices to assume that  $j = 0$ , since the answer is uniform with respect to  $j \geq 0$ ; see Proposition 2.5, (ii). We will consider here the cases  $E_6, F_4, B, C, D$ ; there are 7 such  $\gamma > 0$  for  $E_7$  and 22 for  $E_8$ .

The following are the lists of nonaffine roots  $\gamma$  such that the ends  $\beta = \gamma - \alpha_i, \alpha = \alpha_i$  of  $\lambda(s_{\tilde{\gamma}})$  are *non-movable* under the Coxeter transforms within  $\lambda(s_{\alpha})$  and  $\{\beta, \gamma, \alpha\}$  form an  $A_2$ -triple in the cases of  $F_4, B, C$  (i.e., subject to  $|\alpha_i| = |\gamma|$ ). The bar shows the place of the corresponding  $\alpha_i$ .



**The case of  $E_6$ .** The roots  $\gamma \in R_+$  with non-movable ends:

$$\begin{array}{ccccc} 01\bar{2}10, & 1\bar{2}210, & 012\bar{2}1, & 12\bar{3}21, & 12321. \\ 1 & 1 & 1 & 1 & \bar{2} \end{array}$$

The corresponding  $\{\beta = \gamma - \alpha_i, \gamma, \alpha_i\}$  automatically form a triple.

**The case of  $F_4$ .** The roots  $\gamma \in R_+$  with non-movable ends of  $\lambda(s_\gamma)$  and subject to  $|\alpha_i| = |\gamma|$  are:

$$01\bar{2}1, \quad 1\bar{2}20, \quad 12\bar{3}1, \quad 123\bar{2}, \quad 1\bar{3}42, \quad \bar{2}342.$$

**The case of  $B, C, D_n$ .** Given  $\alpha_i = \epsilon_i - \epsilon_{i+1}$ , the corresponding root  $\gamma$  equals  $\epsilon_{i-1} + \epsilon_i$  for  $i = 2, \dots, n-1$  (it is unique) provided that  $n \geq 3$  and  $i < n-1 \geq 3$  for  $D_n$ . The notation is from [B].

The *Coxeter sequences* of types  $A_2, B_2, G_2$  in reduced decompositions of a given  $\hat{w} \in \widehat{W}$  can be naturally identified with segments of  $\lambda(\hat{w})$  isomorphic to the sequences of positive roots of a rank two systems. By a *Coxeter sequence* in a reduced decompositions of  $\hat{w}$ , we mean a representation  $\tilde{w} = \hat{w}'\tilde{v}\hat{w}''$  subject to  $l(\hat{w}) = l(\hat{w}') + l(\tilde{v}) + l(\hat{w}'')$  such that  $\tilde{v} = (\dots s_i s_j s_i)$  with  $m_{ij} \geq 3$  factors. We will consider *all possible reduced decompositions* of  $\hat{w}$  in the next corollary, i.e., the Coxeter sequences are those that can be made consecutive in at least one reduced decomposition of  $\hat{w}$ .

**Corollary 2.6.** (i) *The triples satisfying one of the conditions (a), (b) or (c) and such that they can be made consecutive in at least one reduced decomposition of  $\hat{w}$  are in one-to-one correspondence with Coxeter sequences of type*

*$A_2$  in case (a) for the root systems  $\tilde{A}, \tilde{D}, \tilde{E}$  or in cases (b), (c),*

*$B_2$  in case (a) for the systems  $\tilde{B}, \tilde{C}, \tilde{F}_4$ , or  $G_2$  for (a) and  $\tilde{G}_2$ .*

(ii) *Let  $\tilde{v}(\tilde{R}_+^0) \subset \lambda(\hat{w})$  for  $\tilde{v} \in \tilde{W}$  and the set of all positive roots  $\tilde{R}_+^0$  of a finite root subsystem  $\tilde{R}^0$  generated over  $\mathbb{Z}$  by a connected subset of  $\{\alpha_0, \dots, \alpha_n\}$ . Then Coxeter transformations can be used to make  $\tilde{v}(\tilde{R}_+^0)$  a (connected) segment in the corresponding  $\lambda(\hat{w})$  if there are no triples  $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$  in  $\lambda(\hat{w}) \cap \tilde{R}_+^0$  that can be extended to a 7-set from (ii) or a 9-set from (iii) (upon the conjugation making  $\tilde{\alpha}$  simple).*

□

**Comment.** (i) The proof below is actually an algorithm of finding such rank two segment(s)  $L$ . We will prove (i,ii,iii) by induction,

assuming that these claims hold for all

$$\{\dots, \tilde{\beta}, \dots, \tilde{\alpha}, \dots\} \text{ in any } \lambda(\widehat{w}) \text{ such that } \ell[\tilde{\beta}, \tilde{\alpha}] = \ell' < \ell,$$

where the  $\ell$ -length  $\ell[\tilde{\beta}, \tilde{\alpha}]$  equals the number of roots in the segment  $[\tilde{\beta}, \tilde{\alpha}] \subset \lambda(\widehat{w})$  including the endpoint. The algorithm below diminishes  $\ell$ . The part of the theorem concerning using  $\tilde{u} \neq \text{id}$  naturally emerges in this procedure too. The last claim from (ii) about using the whole  $\lambda(\widehat{w})$  and its counterpart for (iii) (that was not formulated explicitly) require a somewhat special consideration; it will be omitted.

(ii) Claim (i) of Corollary 2.6 is a straight application of (ii,iii) from the theorem. We will skip the proof of claim (ii) of this corollary; it is not too important in this paper. Generally, there are quite a few situations when we can collect the roots from certain subsets of  $\lambda(\widehat{w})$  together using Coxeter transforms. Claim (ii) gives an example.

Note that long roots that are sums of two short roots are excluded from the theorem; they exist for  $B, C, F$  if the short roots are orthogonal to each other. Such pairs are not needed in Corollary 2.6, (i) for the one-to-one correspondence with the Coxeter transformation of types  $A_2, B_2, G_2$ .

(iii) The theorem can be verified much simpler for  $\tilde{A}_n$  and for the root systems  $\tilde{B}_n, \tilde{C}_n$  where the triples are *not* of type (b). Here either the planar interpretation can be used or the fact that the simple roots have multiplicities no greater than 2 in all positive roots. We will discuss the latter “numerical” approach in detail and give references concerning the planar interpretation.

Let us mention that the 7-sets from (ii) is somewhat simpler to deal with than the 9-sets from (iii); see Lemma 3.1 and especially Lemma 3.2. Also, the case of  $F_4$  is more involved than those of  $B, C$  and requires a special consideration (some details will be omitted).  $\square$

### 3. INDUCTION PROCESS

Let us first clarify property (2.48) for  $\tilde{B}, \tilde{C}, \tilde{F}_4$  under (b).

**3.1. Subsystems  $B_3, C_3$ .** We will describe all possible intersections of  $\lambda$ -sets  $\lambda(\widehat{w})$  with  $R_+^3$  from (ii). The notations from the previous section are used.

**Lemma 3.1.** *Let the triple  $\{\tilde{\alpha}, \tilde{\beta}, \tilde{\alpha} + \tilde{\beta} = \tilde{\gamma}\}$  of type (b) belong to the intersection  $\lambda^0$  of  $\lambda(\hat{w})$  with  $R_+^3$ , the set of the positive roots of type  $B_3$  or  $C_3$  in  $\tilde{R}$ :*

$$\{m\epsilon_i, 1 \leq i \leq 3, \epsilon_i \pm \epsilon_j, i < j \leq 3, \} \subset \tilde{R}_+ \text{ as } m = 1, 2.$$

*Assuming that  $\tilde{\alpha}$  is simple in  $R_+^3$ , the ordered pair  $\{\tilde{\beta}, \tilde{\alpha}\}$  coincides with one of the following pairs:*

$$(3.53) \quad (0) : \{\epsilon_2 - \epsilon_3, \epsilon_1 - \epsilon_2\}, \quad (1) : \{\epsilon_2 + \epsilon_3, \epsilon_1 - \epsilon_2\},$$

$$(3.54) \quad (2) : \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3\}, \quad (3) : \{\epsilon_1 + \epsilon_3, \epsilon_2 - \epsilon_3\}.$$

*The second root (after  $\epsilon_1 - \epsilon_2$ ) in the sequence  $\lambda^0$  can be  $\epsilon_1 - \epsilon_3$  and  $m\epsilon_3$  in cases (0,1); the second root (after  $\epsilon_2 - \epsilon_3$ ) can be  $\epsilon_1 - \epsilon_3$  and  $m\epsilon_2$  in cases (2,3).*

*Case (1a): (1) and the second root is  $\epsilon_1 - \epsilon_3$ . Then the third root is  $m\epsilon_1$  or  $\epsilon_2 - \epsilon_3$  and*

$$\{\tilde{\beta} = \epsilon_2 + \epsilon_3, \epsilon_1 + \epsilon_2, m\epsilon_3, \epsilon_1 + \epsilon_3, m\epsilon_1, \epsilon_1 - \epsilon_3, \tilde{\alpha} = \epsilon_1 - \epsilon_2\} \subset \lambda^0.$$

*Case (2a): (2) and the second root is  $m\epsilon_2$ . Then the third root is  $\epsilon_1 - \epsilon_3$  or  $\epsilon_2 + \epsilon_3$  and*

$$\{\tilde{\beta} = \epsilon_1 - \epsilon_2, m\epsilon_1, \epsilon_1 - \epsilon_3, \epsilon_1 + \epsilon_2, m\epsilon_2, \tilde{\alpha} = \epsilon_2 - \epsilon_3\} \subset \lambda^0.$$

*Case (3a): (3) and  $\epsilon_1 - \epsilon_2 \in \lambda^0$ . Then*

$$\{\tilde{\beta} = \epsilon_1 + \epsilon_3, \epsilon_1 + \epsilon_2, m\epsilon_1, \epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_3, \tilde{\alpha} = \epsilon_2 - \epsilon_3\} \subset \lambda^0.$$

*The second root in  $\lambda^0$  can be  $\epsilon_1 - \epsilon_3$  or  $m\epsilon_2$ .*

*Case (3aa): (3a) and  $m\epsilon_2 \notin \lambda^0$ . Then  $\lambda^0$  is precisely this set. These 6 roots appear in this very order in  $\lambda^0$  modulo the Coxeter transforms in this set.*

*Case (3b): (3) and  $\epsilon_1 - \epsilon_2 \notin \lambda^0$ . Then*

$$\{\tilde{\beta} = \epsilon_1 + \epsilon_3, m\epsilon_1, \epsilon_2 + \epsilon_3, \epsilon_1 + \epsilon_2, \epsilon_1 - \epsilon_3, m\epsilon_2, \tilde{\alpha} = \epsilon_2 - \epsilon_3\} \subset \lambda^0.$$

*Case (3bb): (3b) and  $m\epsilon_3 \notin \lambda(\hat{w})$ . Then these 7 roots constitute  $\lambda^0$  and they appear in this very order in  $\lambda^0$  modulo the Coxeter transforms in this set.*

*Proof.* Theorem 2.1 gives that the intersection  $\lambda^0$  of  $\lambda(\hat{w})$  with the root system  $\tilde{R}^0 = R^3$  is a  $\lambda$ -set with respect to  $R_+^3 = R^3 \cap \tilde{R}_+$ . Using this fact, all claims are straightforward. For instance, all possible orderings of 7 roots from (3bb) are in one-to-one correspondence with

reduced decomposition of  $w = s_2 s_3 s_2 s_1 s_2 s_3 s_2$  in the Weyl group of type  $BC_3$  in the notation from [B]. The Coxeter transformations in  $\lambda^0$  are:

$$\begin{aligned} [\epsilon_2 + \epsilon_3, \epsilon_1 + \epsilon_2, \epsilon_1 - \epsilon_3] &\leftrightarrow [\epsilon_1 - \epsilon_3, \epsilon_1 + \epsilon_2, \epsilon_2 + \epsilon_3], \\ m\epsilon_1 &\leftrightarrow \epsilon_2 + \epsilon_3, \quad \epsilon_1 - \epsilon_3 \leftrightarrow m\epsilon_2. \end{aligned}$$

Let us determine the set  $\lambda^0$  and its ordering in cases (3b,3bb) *directly* via Theorem 2.1; it is an instructional exercise. The condition  $\epsilon_1 - \epsilon_2 \notin \lambda(\widehat{w})$  results in the following:

$$\begin{aligned} (\epsilon_1 - \epsilon_2) + (\epsilon_2 + \epsilon_3) = \widetilde{\beta} &\Rightarrow (\epsilon_2 + \epsilon_3) \text{ is before } \widetilde{\beta}, \\ (\epsilon_1 - \epsilon_2) + 2\epsilon_2 = \widetilde{\gamma} &\Rightarrow 2\epsilon_2 \text{ is before } \widetilde{\gamma}, \\ (\epsilon_1 - \epsilon_3) = (\epsilon_1 - \epsilon_2) + \widetilde{\alpha} &\Rightarrow (\epsilon_1 - \epsilon_3) \text{ is after } \widetilde{\alpha}, \\ 2\epsilon_1 = (\epsilon_1 - \epsilon_2) + \widetilde{\gamma} &\Rightarrow 2\epsilon_1 \text{ is after } \widetilde{\gamma}. \end{aligned}$$

Using the condition  $2\epsilon_3 \notin \lambda(\widehat{w})$ ,

$$\begin{aligned} (\epsilon_2 + \epsilon_3) = \widetilde{\alpha} + 2\epsilon_3 &\Rightarrow (\epsilon_2 + \epsilon_3) \text{ is after } \widetilde{\alpha}, \\ (\epsilon_1 - \epsilon_3) + 2\epsilon_3 = \widetilde{\beta} &\Rightarrow (\epsilon_1 - \epsilon_3) \text{ is before } \widetilde{\beta}. \end{aligned}$$

Then the following relations fix completely the order of all 7 roots between  $\widetilde{\alpha}$  and  $\widetilde{\beta}$  up to the Coxeter transformations in  $\lambda^0$ :

$$\begin{aligned} 2\epsilon_2 = (\epsilon_2 + \epsilon_3) + \widetilde{\alpha} &\Rightarrow 2\epsilon_2 \text{ is after } \widetilde{\alpha}, \text{ since } (\epsilon_2 + \epsilon_3) \text{ is after } \widetilde{\alpha}, \\ 2\epsilon_1 = (\epsilon_1 - \epsilon_3) + \widetilde{\beta} &\Rightarrow 2\epsilon_1 \text{ is before } \widetilde{\beta}, \text{ since } (\epsilon_1 - \epsilon_3) \text{ is before } \widetilde{\beta}. \end{aligned}$$

□

Note that it is always possible to diminish the distance between  $\widetilde{\alpha}, \widetilde{\beta}$  *inside*  $R^3$  unless in case (3bb). One uses the Coxeter transforms of type  $B - C$  in cases (1a) and (2a); the other cases are immediate. It is of course a particular case of the Main Theorem. The cases (0,1,2) can be described in complete detail (similar to (3)) but we do not need it in this paper. An analogous lemma exists for (iii); it is useful when dealing practically with the *admissibility*.

**3.2. Admissible triples.** We will begin with certain reductions and the consideration of classical systems based on relatively straightforward ways of diminishing  $\ell$ . We will call  $\{\widetilde{\alpha} + \widetilde{\beta} = \widetilde{\gamma} \in \lambda(\widehat{w})\}$  satisfying (ii) or (iii) when applicable an **admissible triple**. This notion depends of course on choosing  $\lambda(\widehat{w})$ , not on the triple itself.

The following Lemma simplifies dealing with the admissible triples of type (ii) (its counterpart for (iii) will not be discussed).

**Lemma 3.2.** *In case (3a), the triple  $\{\epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_3\}$  is always admissible unless  $\tilde{R}$  is of type  $\tilde{F}_4$ .*

*Proof.* Let us assume that

$$\epsilon_1 - \epsilon_2 = \epsilon'_1 + \epsilon'_3, \epsilon_1 - \epsilon_3 = \epsilon'_1 + \epsilon'_2, \epsilon_2 - \epsilon_3 = \epsilon'_2 - \epsilon'_3$$

for  $\epsilon'_1, \epsilon'_2, \epsilon'_3$  from  $(R^3)'$  satisfying (3bb). The root system  $\tilde{R}$  can be of type  $\tilde{B}_n$  or  $\tilde{C}_n$ . Let us consider  $\tilde{B}_n$  for the sake of definiteness; thus  $m = 1$ . We will use that any *nonaffine* long root of type  $B$  can be *uniquely* represented as a sums of two *nonaffine* short roots. The last relation gives that

$$\epsilon'_2 = [0, x] - \epsilon_3, \epsilon'_3 = [0, x] - \epsilon_2 \text{ for } x \in \mathbb{N}, \epsilon'_1 = [0, -x] + \epsilon_1.$$

The inequality  $x > 0$  is necessary to make  $\epsilon'_2, \epsilon'_3$  positive (note that it can be insufficient depending on the affine components of these roots). Then  $\epsilon'_1 - \epsilon'_2 = \epsilon_1 + \epsilon_3 + [0, -2x]$ ; it is a positive root and does not belong to  $\lambda(\hat{w})$  (due to (3bb)). However,  $\tilde{\beta} = \epsilon_1 + \epsilon_3$  belongs to  $\lambda$  and therefore all *positive* roots in the form  $\tilde{\beta} - [0, y]$  for  $y \in \mathbb{Z}_+$  must also belong to  $\lambda(\hat{w})$ , a contradiction.  $\square$

**Minimality conditions.** One can suppose that  $\tilde{\alpha} = \alpha_i$  is the *first* root, and  $\tilde{\beta}$  is the *last* in  $\lambda(\hat{w})$ . Note that if  $\tilde{\gamma} = \tilde{\alpha} + \tilde{\beta}$  is the second root (or the last but one) then Proposition 2.2, (iii) can be applied and we may exclude such triples from the consideration.

Since  $\tilde{\alpha}$  is assumed simple in  $R^3, R^4$ , we do not need the cases from (2.47), (2.49) when  $\tilde{\alpha} = \epsilon_1 - \epsilon_3$ . Also we do not need to consider possible inverting of all roots in (2.48). See Lemma 3.1 (where  $\tilde{\alpha}$  was supposed simple).

If there exists at least one simple  $\alpha_j \neq \alpha_i$  in  $\lambda(\hat{w})$ , then we can make  $\alpha_j$  the first in  $\lambda(\hat{w})$  and, therefore, reduce the distance between  $\tilde{\alpha}$  and  $\tilde{\beta}$  and proceed by induction unless  $\alpha_j = \tilde{\beta}$ . In the latter case, the roots  $\tilde{\beta}, \tilde{\gamma}, \tilde{\alpha}$  will be transformed to  $\tilde{\beta}, \tilde{\gamma}, \tilde{\alpha}$  by a chain of Coxeter transformation. Then two of these roots become neighbors during this process somewhere; this case is governed by Proposition 2.2, (iii).

Generally, the theorem for all  $\hat{w}$  and all reduced decompositions is equivalent to the following claim. All reduced decompositions with the  $\lambda$ -sets from  $\tilde{\alpha}$  to  $\tilde{\beta}$  containing  $\tilde{\gamma} = \tilde{\alpha} + \tilde{\beta}$  such that this triple is

admissible (cannot be included in the 7,9-system from (ii,iii) if applicable) have at least two first roots or at least two last roots.

**Comment.** The theorem guarantees that the cardinalities of both sets, the *first roots* and *last roots*, will be greater than one, but considering the first roots here are technically sufficient (if known for all reduced decompositions) to proceed by induction. Note that switching from  $\hat{w}$  to  $\hat{w}^{-1}$  and reversing the order of the corresponding reflections and roots gives that if the number of *first roots* is (always) greater than one then the same holds for the number of *last roots*. For instance, one can impose the condition  $|\tilde{\alpha}| \geq |\tilde{\beta}|$  when/if convenient.  $\square$

Thus  $\alpha_i$  can be assumed a *unique first root* in the sequences  $\lambda(\hat{w})$  for all reduced decompositions of  $\hat{w}$ . This minimality constraint implies (but is not equivalent to) the following conditions for  $\tilde{\alpha} = \alpha_i$ :

- (1)  $\alpha_i$  belongs to any  $\tilde{\delta} \in \lambda(\hat{w})$ , i.e.,  $\tilde{\delta} - \alpha_i \in \tilde{Q}_+ \stackrel{\text{def}}{=} \oplus_{j=0}^n \mathbb{Z}_+ \alpha_j$ ;
- (2) for every  $0 \leq j \neq i$ , if  $\tilde{\delta} - \alpha_j \in \tilde{R}_+$  then  $\tilde{\delta} - \alpha_j \in \lambda(\hat{w})$ .

Similarly, we can suppose that  $\tilde{\beta}$  is a *unique last root* in  $\lambda(\hat{w})$  for all possible reduced decompositions of  $\hat{w}$ ; see Proposition 2.2, (ii). It is equivalent to the conditions:

$$\tilde{\delta} \neq \tilde{\beta} + \tilde{\gamma}' \text{ as } \tilde{\gamma} \in \lambda(\hat{w}) \not\supset \tilde{\gamma}', \quad \tilde{\beta} \neq \tilde{\delta} + \tilde{\gamma}' \text{ as } \tilde{\gamma} \in \lambda(\hat{w}) \ni \tilde{\gamma}'.$$

When constructing the rank two segment  $L$  from the theorem by induction, we will also assume that there are no *smaller admissible triples*  $\tilde{\beta}' + \alpha_i = \tilde{\gamma}'$  in  $\lambda(\hat{w})$  involving  $\alpha_i$  different from  $\tilde{\beta} + \alpha_i = \tilde{\gamma}$ . Otherwise, either we can move  $\alpha_i$  from its first position, which is impossible (see above), or  $L$  is in the very beginning of  $\lambda(\hat{w})$ . In the latter case, the Coxeter transformation in  $L$  moves  $\alpha_i$  from its first position.

Similarly, we will suppose that there are no smaller admissible triples  $\tilde{\beta} + \tilde{\alpha}' = \tilde{\gamma}'$  in  $\lambda(\hat{w})$  involving  $\tilde{\beta}$  different from  $\tilde{\beta} + \alpha_i = \tilde{\gamma}$ .

**3.3. Special cases.** We will begin with some special cases when the theorem (and the corresponding algorithm for finding  $L$ -segments) are relatively simple.

**The case of  $\alpha_i = \alpha_0$ .** The induction argument in this case requires only property (1) above. Let  $\alpha_i = \alpha_0 = [-\vartheta, 1]$ ; recall that  $\vartheta$  is the longest short root. If  $\tilde{\delta} = [\delta, j\nu_\delta] \in \lambda(\hat{w})$  as  $\delta > 0$  then  $\delta = [\delta, 0] \in$

$\lambda(\widehat{w})$ , but the latter does not contain  $\alpha_0$  in its decomposition. Hence,  $\widetilde{\delta} \in \lambda(\widehat{w})$  are always in the form  $\widetilde{\delta} = [-\delta, j]$  is for  $\delta > 0$  when  $i = 0$ .

We set  $\widetilde{\beta} = [-\beta < 0, j\nu_\beta]$ , so  $\widetilde{\gamma} = \widetilde{\beta} + \alpha_0 = [-\beta - \vartheta, j\nu_\beta + 1]$ .

In the simply-laced case,  $\beta + \vartheta$  never belongs to  $R_+$  due to the maximality of  $\vartheta$ .

If there are two different root lengths, then such  $\widetilde{\gamma}$  can exist but must be long due to the definition of  $\vartheta$ ; therefore,  $\beta$  must be short since  $\text{lng} \pm \text{lng} = \text{lng}$ . Therefore, it gives that  $\widetilde{\alpha} + \widetilde{\beta} = \widetilde{\gamma}$  is in the form  $\text{sht} + \text{sht} = \text{lng}$ ; however, this case was excluded from the theorem (see (b)). It concludes the (induction step in the) case  $i = 0$ .

**Systems A, B, C under (a).** Let  $\widetilde{\beta}, \widetilde{\alpha} = \alpha_i, \widetilde{\gamma} = \alpha_i + \widetilde{\beta}$  and the minimality conditions above are imposed. Then a simple “numerical” argument proves claims (i,ii) under assumption (a) for the root systems  $\widetilde{A}, \widetilde{B}, \widetilde{C}$ . It is an instructional example; it also simplifies the process of finding  $L$  in these cases. The *general proof* below will not use this approach (and will include these special cases). The *admissibility conditions* from (ii), (iii) are not applicable to these cases.

The reasoning below works in some other cases, for instance, for (c). We will include the latter in the statement but omit the details.

**Lemma 3.3.** *Let us consider one of the following cases from (2.43) and (2.45):*

$$(3.55) \quad (a) : \widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{G}_2; \quad (c) : A_2 - \text{pure-short roots for } \widetilde{G}_2.$$

*Then (2.48) is not applicable and one can always find the rank two segment  $L$  from the theorem with  $\widetilde{u} = \text{id}$ .*

*Proof.* We may assume that  $i > 0$  and  $|\alpha_i| \geq |\widetilde{\beta}|$ . Thus  $\alpha_i$  is long and  $\widetilde{\beta}$  is short for  $\widetilde{B}, \widetilde{C}, \widetilde{F}, \widetilde{G}$  in case (a). We may also assume that  $\widetilde{\gamma} = \widetilde{\beta} + \alpha_i$  is a unique admissible triple in  $\lambda(\widehat{w})$  involving  $\alpha_i$  as the beginning and that it is unique with  $\widetilde{\beta}$  as the end.

The proof is mainly based on the minimality assumption that all roots  $\widetilde{\delta} \in \lambda(\widehat{w})$  must contain  $\alpha_i$ , i.e., the coefficients of the decomposition of  $\widetilde{\delta} - \alpha_i$  in terms of simple roots  $\alpha_j$  ( $j \geq 0$ ) are all non-negative. In case (b), it is also necessary to check that (2.48) always holds under (3.55); it is not difficult.

There are two possible subcases concerning  $\widetilde{\beta}$ .

*Subcase*  $\tilde{\beta} = [-\beta < 0, j\nu_\beta > 0]$ . Then  $\tilde{\gamma} = [-\gamma, j\nu_\gamma] = [\alpha_i - \beta, j\nu_\beta]$  for  $\gamma = \beta - \alpha_i \in R_+$ . Indeed,  $\beta$  and  $\gamma$  are of the same length because either  $\alpha_i$  is long or  $\alpha_i, \beta, \gamma$  form a root system of type  $A_2$ . The positivity of  $\tilde{\gamma}$  gives that  $\beta - \alpha_i \in R_+$ .

Using that  $\tilde{\gamma} = \tilde{\beta} + \alpha_i$  is assumed to be a unique such triple in  $\lambda(\hat{w})$  involving  $\alpha_i$ , we obtain that  $\tilde{\beta} = [-\beta, \nu_\beta]$  for  $\beta > 0$ . I.e., one can assume that  $j = 1$ . The latter assumption and the positivity of  $\tilde{\beta}$  implies that the multiplicity of  $\alpha_0 = [-\vartheta, 1]$  in  $\tilde{\beta}$  is one for short  $\beta$  and two if it is long.

If  $\beta$  is short, then  $\vartheta - \beta$  must contain  $\alpha_i$  (in the decomposition in terms of  $\{\alpha_1, \dots, \alpha_n\}$ ). If  $\beta$  is long, then  $2\vartheta - \beta$  must contain  $\alpha_i$ . Combining it with  $\beta - \alpha_i \in R_+$ , we conclude that the multiplicity of  $\alpha_i$  in  $\vartheta$  or  $2\vartheta$  correspondingly must be at least 2. This is *impossible*. Thus one can diminish the  $\ell$ -length and proceed by induction in the cases under consideration.

*Subcase*  $\tilde{\beta} = [\beta > 0, j\nu_\beta \geq 0]$ . Using the minimality, we can assume that  $j = 0$  and  $\tilde{\beta} = \beta$ . Recall that  $\alpha_i$  is contained in  $\beta \in R_+$  and  $\tilde{\gamma} = \gamma = \alpha_i + \beta$ . In case (a),  $\alpha_i$  is long,  $\beta$  is short, and  $\gamma$  is short. Then the coefficient of  $\alpha_i$  in  $\gamma$  can not be greater than 1. Therefore we can diminish the  $\ell$ -length between  $\tilde{\alpha}$  and  $\tilde{\beta}$  and proceed by induction.

Note that formally this argument can be used in the cases

$$\tilde{B}(\text{sht} + \text{sht}) \text{ or } \tilde{C}(\text{lng} + \text{lng}),$$

that were excluded from the theorem since such combinations do not lead to  $A_2$ -triples.  $\square$

**Comment.** (i) In the cases  $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ , one can use the plane interpretation of the reduced decompositions from [C2]; it exists for  $G_2$  too. For instance, it is geometrically obvious that the roots

$$\tilde{\alpha} = \epsilon_2 - \epsilon_3, \tilde{\beta} = \epsilon_1 + \epsilon_3, \tilde{\gamma} = \tilde{\alpha} + \tilde{\beta} = \epsilon_1 + \epsilon_2,$$

from (2.48) cannot be collected together. Algebraically, the 7-set there corresponds to the reduced nonaffine decomposition  $w = s_2 s_3 s_2 s_1 s_2 s_3 s_2$ , where  $s_1$  transposes  $\epsilon_1$  and  $\epsilon_2$ ,  $s_2$  is the transposition  $\epsilon_2 \leftrightarrow \epsilon_3$ , and  $s_3$  is the reflection  $\epsilon_3 \mapsto -\epsilon_3$ . It is analogous for the 9-root set of type  $D_4$  from (2.51).

(ii) Geometrically, one plots 3 lines in the half-plane  $\{(x, y) \in \mathbb{R}^2, y > 0\}$  subject to the reflection in the  $x$ -axis. The intersection and reflection points give a reduced decomposition of  $w$ ; the corresponding angles



constitute  $\lambda(w)$  (the reflection angle must be multiplied by 2 as  $m = 2$ ). Here  $\epsilon_i$  is interpreted as the (initial, before the reflection) angle of line  $i$ . They are counted with respect to the intersection points with  $x = b$  from the highest  $y$  down to the  $x$ -axis for sufficiently large  $b$  and then the intersection and reflection points are examined as  $a < x < b$ , provided that their ultimate transformation from  $x = b$  to  $x = a$  is  $w$ .

(iii) For  $w$  from (i), the first two lines have “almost” coinciding nonzero angles  $\epsilon_1, \epsilon_2$ ;  $\epsilon_3$  is “almost” zero and intersects the first two before and after the reflection points. Generally, one takes  $n$  lines for  $B_n, C_n$  and take  $w$  for  $(n-1)$  “almost” parallel non-horizontal lines and one horizontal line. Then the first and the last angles (and their sum) form a *non-admissible* triple; the  $\ell$ -distance between them cannot be diminished by Coxeter transformations, which is obvious geometrically. The theorem claims that this example is, in a sense, the only *obstacle* for collecting triples in rank two segments. It is analogous for  $D_n$ .

(iv) The affine variant of this interpretation requires a portion of  $\mathbb{R}^2$  trapped between *two reflection lines*; it covers the root system  $C^\vee C$  including affine  $B, C, D$ . The affine  $A_n$ -system is described by  $n+1$  lines on a *cylinder* (with the periodic  $y$ -coordinate). This interpretation is helpful, but the algebraic approach is important even for the classical root systems; the above simple proof for  $\tilde{A}, \tilde{B}, \tilde{C}$  under (a) is a clear demonstration. Note that considering triples (there are many similar situations) is the key for the technique of intertwiners.  $\square$

**3.4. Uniform construction.** Let  $\tilde{\beta}, \tilde{\alpha} = \alpha_i, \tilde{\gamma} = \alpha_i + \tilde{\beta}$  is an *admissible triple*; one of the conditions (a), (b), (c) must hold. We impose the minimality conditions (1) and (2) above, assuming that  $\alpha_i$  ( $0 \leq i \leq n$ ) is a unique simple root in  $\lambda(\hat{w})$

We will also assume later that neither  $\tilde{\alpha}$  nor  $\tilde{\beta}$  belong to *smaller admissible triples*. The condition  $|\alpha_i| \geq |\tilde{\beta}|$  will be imposed (unless stated otherwise); it can be always provided using the inversion  $\hat{w} \mapsto \hat{w}^{-1}$  if necessary. It results in  $s_i(\tilde{\beta}) = \tilde{\gamma}$ ,  $s_i(\tilde{\gamma}) = \tilde{\beta}$ .

We will examine the second root in  $\lambda(\hat{w})$  that is  $s_i(\alpha_j) = \alpha_j + \mu\alpha_i$  for  $\hat{w} = \cdots s_j s_i$  as  $0 \leq j \neq i$  and  $\mu \in \mathbb{N}$  (if  $\mu = 0$  then  $\alpha_j \in \lambda(\hat{w})$ ).

Note that  $\mu > 1$  only in the non-simply-laced case and if and only if  $\alpha_i$  is short and  $\alpha_j$  is long. Then  $\tilde{\beta}$  has to be short, due to  $|\alpha_i| \geq |\tilde{\beta}|$ , that implies in its turn that  $\tilde{\gamma}$  is short.

Let  $\tilde{\alpha}' \stackrel{\text{def}}{=} s_j(\alpha_i) = \alpha_i + \mu' \alpha_j$ , where  $\mu' > 1$  occurs only if  $\alpha_j$  and  $\alpha_i$  have different lengths and  $|\alpha_i| > |\alpha_j|$ .

Note that  $\mu\mu' = m_{ij} = 1, 2, 3$ , where  $m_{ij}$  is the number of laces between  $\alpha_i$  and  $\alpha_j$  in the affine Dynkin diagram, and at least one of  $\mu, \mu'$  equals 1, namely,  $\mu = 1$  if  $|\alpha_i| > |\alpha_j|$ ,  $\mu' = 1$  if  $|\alpha_j| > |\alpha_i|$ ;  $\mu = 1 = \mu'$  if  $|\alpha_i| = |\alpha_j|$ .

One has  $\tilde{\beta}' \stackrel{\text{def}}{=} s_j(\tilde{\beta}) = \tilde{\beta} + p\alpha_j$ , where  $-\mu' \leq p \leq \mu'$ . If  $|\tilde{\beta}| = |\alpha_i|$  then  $p = -\mu', 0, \mu'$ ; if  $|\tilde{\beta}| < |\alpha_i|$  then  $|\tilde{\beta}| \leq |\alpha_j|$  and  $p = -1, 0, 1$ .

If  $p > 0$ , then  $\tilde{\gamma}' \stackrel{\text{def}}{=} s_j(\tilde{\gamma}) = \tilde{\gamma} + (\mu' + p)\alpha_j$ . If  $|\tilde{\beta}| = |\alpha_i|$ , then  $|\tilde{\gamma}| = |\alpha_i|$  and  $(\mu' + p)$  must be no greater than 1 (as for  $\alpha_i$ ), which is impossible. If these lengths are different, then  $\tilde{\gamma}$  has to be short since  $\text{sht} + \text{lng}$  is  $\text{sht}$  and  $(\mu' + p) \geq 1$ ; this is impossible too.

**Lemma 3.4.** *Let  $m_{ij} = 1$ , which implies  $\mu = 1 = \mu'$ . Then  $\tilde{\alpha}' = \alpha_i + \alpha_j \in \lambda(\hat{w})$  and  $\tilde{\beta}' = s_j(\tilde{\beta}) = \tilde{\beta} + p\alpha_j$  for  $p \leq 0$ . In this case, the triple  $\{\tilde{\beta}', \tilde{\gamma}' = \tilde{\alpha}' + \tilde{\beta}', \tilde{\alpha}'\}$  belongs to  $\lambda(\hat{w})$ . Moreover, the triple  $\{\tilde{\beta}', \tilde{\gamma}' = \tilde{\alpha}' + \tilde{\beta}', \tilde{\alpha}'\}$  is admissible if  $\{\tilde{\beta}, \tilde{\gamma} = \tilde{\alpha} + \tilde{\beta}, \tilde{\alpha}\}$  is admissible.*

*Proof.* Using  $\mu = 1 = \mu'$ ,  $\tilde{\alpha}' = s_i(\alpha_j) \in \lambda(\hat{w})$ . Since  $p \leq 0$  (see above) and  $\alpha_j \notin \lambda(\hat{w})$ ,  $\tilde{\beta}' = \tilde{\beta} + p\alpha_j$  and  $\tilde{\beta}'$  has to be in  $\lambda(\hat{w})$  (property (2) from the minimality conditions). Therefore  $\tilde{\gamma}'$  belongs to this set too.

Let us suppose that the triple  $\tilde{\alpha}' + \tilde{\beta}' = \tilde{\gamma}'$  can be extended to the set of 7 roots in the segment  $[\tilde{\beta}', \tilde{\alpha}']$  of type (2.48) as  $\tilde{\alpha}' = \epsilon'_2 - \epsilon'_3$ ,  $\tilde{\beta}' = \epsilon'_1 + \epsilon'_3$ ,  $\tilde{\gamma}' = \epsilon'_1 + \epsilon'_2$ .

The set of roots  $(R^3)'_+ \stackrel{\text{def}}{=} \{m\epsilon'_i, \epsilon'_i \pm \epsilon'_j, i < j\}$  is positive in  $\tilde{R}$  (by construction). Since  $\tilde{\alpha}'$  is simple in  $(R^3)'_+$ , the root  $\alpha_j$  does not belong to  $(R^3)'_+$ ; indeed,  $\tilde{\alpha} = \tilde{\alpha}' - \alpha_j > 0$  in  $\tilde{R}$  and therefore in  $R^3$ . Therefore the image of  $(R^3)'_+$  under  $s_j$  is positive in  $\tilde{R}$ :

$$R^3_+ \stackrel{\text{def}}{=} \{m\epsilon_i, \epsilon_i \pm \epsilon_j, i < j\} \subset \tilde{R}_+ \quad \text{for } \epsilon_k = s_j(\epsilon'_k), k = 1, 2, 3.$$

The roots

$$\tilde{\alpha} = \epsilon_2 - \epsilon_3, \tilde{\beta} = \epsilon_1 + \epsilon_3, \tilde{\gamma} = \epsilon_1 + \epsilon_2$$

belong to  $R^3_+$  by construction; Lemma 3.1 describes all possibilities. Using Lemma 3.2 (and its counterpart for  $\tilde{D}_n$  and  $\tilde{C}_{n \geq 4}$  with  $R^4$  subsystems) we can conclude the proof for the classical systems. The case

of  $\tilde{F}_4$  requires a special consideration. Without using this lemma we can proceed as follows.

We combine  $(R^3)'$  and  $R^3$  in a root subsystem  $\hat{R}^4 \subset \tilde{R}$  of type  $B_4$  or  $C_4$ , defined as an intersection of  $\tilde{R}$  with the  $Q$ -span of  $R^3$  and  $\alpha_j$ . The latter root does not belong to  $R^3$  or  $(R^3)'$  and is simple in  $R^4$ . The root  $\alpha_j$ , the set of roots

$$(3.56) \quad \{\tilde{\beta}, \tilde{\beta}' = \tilde{\beta} + p\alpha_j, \tilde{\gamma}, \tilde{\gamma}', \tilde{\alpha}' = \alpha_i + \alpha_j, \tilde{\alpha} = \alpha_i\} \subset \lambda(\hat{w}),$$

and also the images of the remaining 4 roots from the 7-set in  $(R_+^3)'$  belong to  $\hat{R}_+^4$ . Generally, the latter system can be an affine extension of  $R^3$  with  $\alpha_j$  being the affine simple root, but this case can be excluded from the consideration.

Then we prove the lemma for  $\hat{R}^4$ , i.e., deduce that  $\{\tilde{\beta}', \tilde{\gamma}', \tilde{\alpha}'\}$  is admissible from the admissibility of  $\{\tilde{\beta}, \tilde{\gamma}, \tilde{\alpha}\}$ . Here we intersect  $\lambda(\hat{w})$  with  $R^4$ ; the intersection is a  $\lambda$ -set in the latter. This method remains essentially the same in the simply-laced case (and for  $\tilde{C}_{n \geq 4}$ ), but the new subsystem will be  $\hat{R}^5$  of type  $D_5$  instead of  $R^4$ .

The logic of this approach becomes more transparent if one first proves the Main Theorem for the nonaffine root systems of type  $B_4, C_4$  and  $D_5$ . Then the initial admissible triple  $\{\tilde{\beta}, \tilde{\gamma}, \tilde{\alpha}\}$  can be collected in a rank two  $L$ -segment in  $\hat{R}^4$  (or  $\hat{R}^5$ ). However,  $\alpha_j \notin \lambda(\hat{w})$  and the order of roots in (3.56) must remain the same if the order of  $\{\tilde{\beta}, \tilde{\gamma}, \tilde{\alpha}\}$  is unchanged upon the Coxeter transforms. Therefore, in process of collecting  $\{\tilde{\beta}, \tilde{\gamma}, \tilde{\alpha}\}$  in a  $L$ -segment of rank two in  $R^4$  (or  $R^5$ ), we will automatically make  $\{\tilde{\beta}', \tilde{\gamma}', \tilde{\alpha}'\}$  collected together (in a  $L$ -segment). This gives the admissibility of  $\{\tilde{\beta}', \tilde{\gamma}', \tilde{\alpha}'\}$ . We will omit proving Main Theorem 2.4 in the non-affine cases  $B_4, C_4, D_5$ , and the details of the consideration of  $\tilde{F}_4$ .  $\square$

**3.5. The case  $m_{ij} > 1$ .** We will first consider the case when  $s_j$  acts trivially on  $\tilde{\beta}$ .

**Case  $p = 0$ .** Then  $\tilde{\beta} = \tilde{\beta}'$  and  $\alpha_i$  and  $\tilde{\gamma}$  must be of the same length for short  $\alpha_j$  due to  $s_j(\tilde{\gamma} - \alpha_i) = 0$  and  $(\tilde{\gamma}, \alpha_j) = (\alpha_i, \alpha_j)$ . Recall that  $s_i(\tilde{\delta}), s_j(\tilde{\delta}), s_i s_j(\tilde{\delta})$  are positive for any  $\tilde{\delta} \in \lambda(\hat{w})$  after  $s_j$ , i.e., for all roots excluding the first two roots,  $\alpha_i$  and  $\alpha_j + \mu\alpha_i$ . Note that the

positivity of  $s_i(\tilde{\delta})$  and  $s_j(\tilde{\delta})$  readily follows from the fact that  $\alpha_i \neq \tilde{\delta} \neq \alpha_j$ .

If  $m_{ij} = 1$  then  $\{\tilde{\beta}, \tilde{\gamma}', \tilde{\alpha}'\}$  is admissible due to Lemma 3.4 and we can change the position of  $\tilde{\beta}$  in the segment  $[\tilde{\beta}, \tilde{\alpha}] \subset \lambda(\hat{w})$ , which is impossible due to the minimality conditions. Similarly, we can assume that  $\alpha_i$  and  $\tilde{\gamma}$  are the same length for any  $\alpha_j$ ; otherwise the admissibility is automatic for unequal lengths (see case (a)). Thus, it suffices to consider only the case  $m_{ij} > 1$ , where  $\tilde{R}$  can be of types  $\tilde{B}, \tilde{C}, \tilde{F}_4, \tilde{G}_2$ .

If  $m_{ij} > 1$  for these root systems, then the roots

$$\alpha^1 = \tilde{\beta} = \epsilon_1 - \epsilon_2, \quad \alpha^2 = \alpha_i = \epsilon_2 - \epsilon_3, \quad \alpha^3 = \alpha_j = m\epsilon_3$$

are simple roots in  $\tilde{R}^0$ , which is defined as the intersection of  $\tilde{R}$  with the  $\mathbb{Z}$ -span of  $\alpha_i, \alpha_j, \tilde{\gamma}$  with the positivity with respect to  $\tilde{R}_+^0 = \tilde{R}_+ \cap \tilde{R}^0$ . It is a *finite* root system of rank 3 unless in the  $G_2$  case (we leave this case to the reader). The set  $\lambda(\hat{w})$  does not contain  $m\epsilon_3$  and therefore must contain  $m\epsilon_2$ . It is the case (2a) of Lemma 3.1, the second root is  $m\epsilon_2$ . One can inverse the order in the subset  $\{\epsilon_1 - \epsilon_2, m\epsilon_1, \epsilon_1 + \epsilon_2, m\epsilon_2\}$  in  $\lambda(\hat{w})$  using the induction statement in case (a) of the theorem (here there is no problem with (2.48)).

After moving  $m\epsilon_2$  from its first position inside the segment  $[\epsilon_1 - \epsilon_2, m\epsilon_2] \in \lambda(\hat{w})$ , the root  $\epsilon_1 - \epsilon_3$  automatically becomes the second in  $\lambda(\hat{w})$  (right after  $\alpha_i = \epsilon_2 - \epsilon_3$ ). Then part (iii) of Proposition 2.2 guarantees that we can make the triple  $\{\epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_3\}$  a connected sequence in  $\lambda(\hat{w})$  for a proper reduced decomposition of  $\hat{w}$ . Then we can move  $\alpha_i$  from its first position. However the latter contradicts to the minimality conditions and therefore concludes the consideration of the case  $p = 0$ .

**Comment.** The consideration of  $p = 0, m_{ij} > 1$  for  $\tilde{B}, \tilde{C}, \tilde{G}_2$  can be concluded following the special cases  $\tilde{A}, \tilde{B}, \tilde{C}$  under (a) considered above. The roots  $\alpha^1 = \tilde{\beta}, \alpha^2 = \alpha_i, \alpha^3 = \alpha_j$  are *simple roots* of  $\tilde{R}^0$ , the intersection of  $\tilde{R}$  with the  $\mathbb{Z}$ -span of  $\alpha_i, \alpha_j, \tilde{\gamma}$  with respect to  $\tilde{R}_+^0 = \tilde{R}_+ \cap \tilde{R}^0$ . It is a *finite* root system of rank 3 unless in the  $G_2$  case (we leave this case to the reader). The intersection  $\lambda(\hat{w}) \cap \tilde{R}$  is a  $\lambda$ -set in  $\tilde{R}_+^0$ . For the sake of simplicity, let the triple and  $\alpha_j$  be nonaffine. Then  $\tilde{\gamma} - 2\alpha_i \geq 0$  and  $s_i s_j(\tilde{\gamma}) = \tilde{\gamma} + (m_{ij} - 1)\alpha_i + \mu'\alpha_j$  must contain

$(m_{ij} + 1)\alpha_i$ , which is impossible. Similar approach can be applied to  $\widetilde{F}_4$ .  $\square$

*Thus, the case  $p < 0$  is sufficient to consider.*

To simplify the exposition we exclude the case  $\{\mu' > 1, p = -1\}$ . It can be done since  $|\alpha_j| < |\alpha_i|$  in this case, i.e.,  $\alpha_j$  is short. Therefore  $|\widetilde{\beta}| = |\alpha_j|$  due to  $\widetilde{\beta}' = s_j(\widetilde{\beta}) = \widetilde{\beta} + p\alpha_j = \widetilde{\beta} + \alpha_j$ . This gives  $|\widetilde{\beta}| < |\alpha_i|$ . With this inequality, no admissibility condition is needed and the proof of the theorem is straightforward.

Thus, it suffices to assume that  $|\widetilde{\beta}| = |\alpha_i|$  and  $p = -\mu'$ . We have:

$$\widetilde{\alpha}' = s_j(\alpha_i) = \alpha_i + \mu' \alpha_j, \quad \widetilde{\beta}' = s_j(\widetilde{\beta}) = \widetilde{\beta} - \mu' \alpha_j, \quad s_j(\widetilde{\gamma}) = \widetilde{\gamma}$$

in this case. Since  $\widetilde{\beta}'$  is a positive root, it must belong to  $\lambda(\widehat{w})$  (the minimality property (2) above).

The element  $\widetilde{\alpha}'$  is a sum of positive roots and therefore belongs to  $\widetilde{R}_+$ . However, generally, it can be apart from  $\lambda(\widehat{w})$ . The following lemma addresses this possibility.

**Lemma 3.5.** *We assume that  $\alpha_i, \widetilde{\gamma}, \widetilde{\beta}$  are of equal lengths and constitute an admissible triple in  $\lambda(\widehat{w})$ . The minimality conditions are imposed, in particular,  $\alpha_i$  is a unique simple root in  $\lambda(\widehat{w})$ ; recall that  $\alpha_i$  and  $s_i(\alpha_j) = \alpha_j + \mu\alpha_i$  are the first and the second root in this set. The claim is that  $\widetilde{\alpha}' = s_j(\alpha_i) = \alpha_i + \mu' \alpha_j$  belongs to  $\lambda(\widehat{w})$ .*

*Proof.* The inclusion  $\widetilde{\alpha}' \in \lambda(\widehat{w})$  can be wrong only when  $\mu \neq \mu'$ , i.e., as  $m_{ij} > 1$  (for  $\widetilde{B}, \widetilde{C}, \widetilde{F}_4, \widetilde{G}_2$ ). Also, either  $\alpha_j$  must be long and  $\alpha_i, \widetilde{\beta}, \widetilde{\gamma}$  short, or it can be the other way round, with short  $\alpha_j$  and long  $\alpha_i, \widetilde{\beta}, \widetilde{\gamma}$ . The latter variant is impossible for  $\widetilde{G}_2$ , and both variants are completely analogous for  $\widetilde{B}, \widetilde{C}, \widetilde{F}_4$ . We will consider here only the first one, assuming that

$$\mu' = 1 = -p, \quad \mu > 1, \quad \widetilde{\alpha}' = \alpha_i + \alpha_j, \quad |\alpha_j| > |\alpha_i| = |\widetilde{\beta}| = |\widetilde{\gamma}|.$$

Since  $l(\widehat{w}) = l(\widehat{w}s_i s_j) + l(s_j s_i)$ , the following element must be a positive root in  $\widetilde{R}$ :

$$s_i s_j(\widetilde{\beta}) = s_i(\widetilde{\beta} - \alpha_j) = (\widetilde{\beta} + \alpha_i) - (\alpha_j + \mu\alpha_i) = \widetilde{\beta}' - (\mu - 1)\alpha_i.$$

Therefore  $\widetilde{\beta}'' \stackrel{\text{def}}{=} \widetilde{\beta} - (\alpha_i + \alpha_j)$  is a positive root too; also  $(\widetilde{\beta}'', \alpha_j) = 0$ .

Setting  $\alpha^1 = \widetilde{\beta}'', \alpha^2 = \alpha_i, \alpha^3 = \alpha_j$ , let  $\widetilde{R}^0$  be a root system  $\mathbb{Z}$ -generated by these roots in  $\widetilde{R}$ . Then they are simple roots in  $\widetilde{R}_+^0 =$

$\tilde{R}^0 \cap \tilde{R}_+$ . The system  $\tilde{R}^0$  is of type  $C_3$  as  $\mu = 2$ . In the case of  $G_2$  ( $\mu = 3$ ), one also has that  $(\tilde{\beta}'', \alpha_i) = 0$ . It implies that  $\tilde{\beta}''$  can not be a (real) root. Therefore  $\mu = 3$  can be excluded from further considerations; we will assume that  $\mu = 2$ .

Since  $\lambda(\hat{w}) \ni \tilde{\beta} = \tilde{\beta}'' + (\alpha_i + \alpha_j)$ , then  $\tilde{\beta}'' = \tilde{\beta}' - \alpha_i \in \lambda(\hat{w})$  if  $\alpha_i + \alpha_j \notin \lambda(\hat{w})$  (i.e., if the claim of the lemma does not hold). Then we proceed as follows. Setting

$$\alpha^1 = \tilde{\beta}'' = \beta - \alpha_i - \alpha_j = \epsilon_1 - \epsilon_2, \alpha^2 = \alpha_i = \epsilon_2 - \epsilon_3, \alpha^3 = \alpha_j = 2\epsilon_3,$$

the intersection  $\lambda(\hat{w}) \cap \tilde{R}_+^0$  is a  $\lambda$ -sequence in  $\tilde{R}_+^0$ , and it is not difficult to check that it consists of the following roots in the order of appearance:

(3.57)

$$\{\tilde{\beta} = \epsilon_1 + \epsilon_3, \tilde{\beta} - \alpha_i - \alpha_j = \epsilon_1 - \epsilon_2, 2\tilde{\beta} - \alpha_j = 2\epsilon_1, \tilde{\gamma} = \epsilon_1 + \epsilon_2, \\ \tilde{\beta} - \alpha_j = \epsilon_1 - \epsilon_3, \alpha_j + 2\alpha_i = 2\epsilon_2, \alpha_i = \epsilon_2 - \epsilon_3\}.$$

One can also use claim (ii) of the theorem for  $\tilde{R}^0$  instead of  $\tilde{R}$  here. It gives that  $\alpha_i, \tilde{\beta}$  can be transformed to a rank two segment  $L^0$  of type  $A_2$  inside  $\lambda(\hat{w}) \cap \tilde{R}_+^0$ . Indeed,  $\tilde{\beta}''$  is *simple* in  $\tilde{R}_+^0$  and one can make it the first root in this sequence instead of  $\alpha_i$ . The set (3.57) is from case (2a) of Lemma 3.1 and we can use Lemma 3.2 to come to a contradiction with the minimality conditions.

We obtained that  $s_j(\alpha_i) = \alpha_i + \alpha_j \in \lambda(\hat{w})$  for long  $\alpha_j$  and short  $\alpha_i, \tilde{\beta}, \tilde{\gamma}$ ; the case of short  $\alpha_j$  and long  $\alpha_i, \tilde{\beta}, \tilde{\gamma}$  is analogous.  $\square$

**3.6. Concluding the induction.** Thus, the case  $\mu = \mu' = 1$  is sufficient for obtaining a transformation to the rank two  $L$  from (i) or (ii,iii) containing  $\alpha_i, \tilde{\beta}, \tilde{\gamma}$ . In this case, the admissible triples constructed above for  $\tilde{\gamma}$  with the beginning at  $\alpha_i + \alpha_j$  and end at  $\tilde{\beta} - \alpha_j$  lead to finding a simple root in  $\lambda(\hat{w})$  different from  $\alpha_i$ . Let us demonstrate it.

We will omit the case of  $\tilde{A}_n$  for the sake of uniformity; the theorem has been already checked for this root system. Then the affine Dynkin diagram  $\tilde{\Gamma}$  is a tree.

Thus,  $\mu = \mu' = 1$  and  $\alpha_i + \alpha_j$  is the *second* root after  $\alpha_i$  in  $\lambda(\hat{w})$ . Using the admissible triple constructed above, the induction hypothesis makes it possible to move  $\alpha_i + \alpha_j$  from its second position in  $\lambda(\hat{w})$  ( $\alpha_i$

remains untouched). Therefore a simple  $s_k \neq s_j$  with  $\alpha_k$  of the same length as  $\alpha_i$  must exist such that it can be made the *second* after  $s_i$  upon a suitable transformation of the initial reduced decomposition of  $\widehat{w}$  (without touching  $s_i$ ). Such  $\alpha_k$  must be connected with  $\alpha_i$  by a link in  $\widetilde{\Gamma}$  and therefore must be orthogonal to  $\alpha_j$ ; so  $s_j s_k = s_k s_j$ .

We conclude that all *first roots* in  $\lambda(\widehat{w}')$  for  $\widehat{w}' = \widehat{w} s_i$  are pairwise orthogonal and there exists a reduced decomposition of  $\widehat{w}$  in the form  $\widehat{w} = \cdots s_k s_j s_i = \cdots s_j s_k s_i$ . Note that the corresponding  $\lambda$ -set contains the following roots:

$$\widetilde{\beta}, \dots, \widetilde{\beta} - \alpha_j, \dots, \widetilde{\beta} - \alpha_k, \dots, \widetilde{\gamma}, \dots, \alpha_i + \alpha_k, \alpha_i + \alpha_j, \alpha_i;$$

here we can transpose  $\alpha_i + \alpha_k, \alpha_i + \alpha_j$  and the order of appearance of  $\widetilde{\beta} - \alpha_j, \widetilde{\beta} - \alpha_k$  can be different.

Let us exclude for a while the cases  $\widetilde{D}, \widetilde{E}, \widetilde{B}$  (when  $\widetilde{\Gamma}$  is not a segment). Then the number of *first roots* in  $\lambda(\widehat{w}')$ , is exactly 2, namely, they are  $\alpha_j$  and  $\alpha_k$ .

The  $\lambda$ -set of  $\widehat{w}'' = \widehat{w} s_i s_k$  contains an admissible triple that begins with  $\alpha_j$ ; explicitly, it is

$$\{s_k s_i(\widetilde{\beta} - \alpha_j) = s_k(\widetilde{\beta} - \alpha_j) = \widetilde{\beta} - \alpha_j - \alpha_k, \dots, s_k s_i(\widetilde{\gamma}) = \widetilde{\beta} - \alpha_k, \dots, \alpha_j\}.$$

Therefore it must have at least one *first root*  $\alpha_l \neq \alpha_j$ . If  $|\alpha_l| = |\alpha_i|$  then this root cannot be  $\alpha_i$ , since otherwise  $s_i s_k s_i = s_k s_i s_k$  and it would make  $\alpha_k$  another *first root* of  $\widehat{w}$ . So it must be connected by a link with  $\alpha_k$  in  $\widetilde{\Gamma}$ . Indeed, otherwise it would be the third *first root* in  $\lambda(\widehat{w}')$ , which is impossible. This means that it cannot be connected with  $\alpha_j$  ( $\widetilde{\Gamma}$  is a tree) and we have again the orthogonality of all *first roots* in  $\lambda(\widehat{w}'')$ . The connection picture is  $\alpha_l \longleftrightarrow \alpha_k \longleftrightarrow \alpha_i \longleftrightarrow \alpha_j$ .

We continue this process for  $\widehat{w}''' = \widehat{w} s_i s_k s_l$ ; the reflection  $s_j$  can be made again the beginning of its reduced decomposition, respectively,  $\alpha_j$  can be assumed to be the first in  $\lambda(\widehat{w}''')$ . The latter sequence contains the triple with  $\alpha_j$  as its first endpoint; this triple is the image of  $\{\widetilde{\beta} - \alpha_j, \widetilde{\gamma}, \alpha_i + \alpha_j\}$  under  $s_l s_k s_i$ . Once again, another *first root*  $\alpha_m \neq \alpha_j$  must exist for  $\widehat{w}'''$ . Continuing to assume that  $|\alpha_l| = |\alpha_i|$ , it cannot be  $\alpha_i$  (otherwise we could move  $s_i$  through  $s_l$  and use the Coxeter relation for  $s_i s_k s_i$ ) and it must be connected with  $\alpha_l$  or with  $\alpha_k$ . Hence,  $\alpha_m$  cannot be connected with  $\alpha_j$ , so it is orthogonal to  $\alpha_j$ . We can represent  $\widehat{w}''' = \cdots s_j s_m$  and then continue with  $\widehat{w}'''' = \widehat{w}''' s_m$ .

Eventually, we come to an element  $\widehat{w}s_i s_k s_l s_m \cdots$  of length 3 and with the  $\lambda$ -set that is a pure triple with  $\alpha_j$  as the beginning. However such a set does not have pairwise orthogonal *first roots*. This contradiction proves the existence of the rank two  $L$  from (i,ii) of the theorem (provided the admissibility).

Recall, that it was proven for  $\widetilde{\Gamma}$  that is a segment and under the assumption that we can always find the roots  $\alpha_l, \alpha_m, \dots$  of the same length as that for  $\alpha_i$  (and in the non-simply-laced case).

If a root in the latter sequence of *first roots* has the length different from  $|\alpha_i|$  (it can be only an endpoint of  $\widetilde{\Gamma}$  unless in the  $\widetilde{F}_4$ -case), then we use  $k$  instead of  $j$  and add a simple root  $\alpha_{l'}$  connected with  $\alpha_j$ ; then  $\alpha_{m'}$  is connected with  $\alpha_{l'}$  and so on. All their lengths will be coinciding with  $|\alpha_i|$  when moving in this direction (there is only one double link in  $\widetilde{\Gamma}$ ).

A similar reasoning can be applied in the simply-laced case when  $\widetilde{\Gamma}$  is not a segment (then one can, generally, proceed using three directions).

To complete the proof of the theorem we need to check that if  $\widetilde{\alpha}$  is before  $\widetilde{\beta}$  in  $\lambda(\widehat{w})$  and this set contains exactly the 7 roots listed in (2.48) upon the intersection with the  $\mathbb{Z}$ -span of these seven roots, then

- 1) all these 7 roots belong to the segment  $[\widetilde{\beta}, \widetilde{\alpha}]$ ,
- 2) their appearance in  $[\widetilde{\beta}, \widetilde{\alpha}]$  is as in (2.48).

The latter holds modulo the Coxeter transformations inside this set. It readily results from Lemma 3.1, case (3). The analogous claim for (2.51) is equally simple.  $\square$

The following corollary, a variant of Lemma 1.2, demonstrates how the theorem can be used in an important particular case of reflections.

**Corollary 3.6.** *Given  $\widetilde{\gamma} = [\gamma, \nu_\gamma j] \in \widetilde{R}_+$ , let  $\widetilde{\alpha} \in \lambda(s_{\widetilde{\gamma}})$  and  $\nu_\alpha = \nu_\gamma$ . Then  $\widetilde{\beta} = -s_{\widetilde{\gamma}}(\widetilde{\alpha}) = \widetilde{\gamma} - \widetilde{\alpha}$  belongs to  $\lambda(s_{\widetilde{\gamma}})$  due to (1.20).*

(i) *Given a reduced decomposition of  $s_{\widetilde{\gamma}}$ , the triple  $\{\widetilde{\beta}, \widetilde{\gamma}, \widetilde{\alpha}\}$  for simple  $\widetilde{\alpha}$  can be made consecutive in  $\lambda(s_{\widetilde{\gamma}})$  if and only if  $\widetilde{\gamma} \neq \theta'$  in any root subsystem  $R' \subset \widetilde{R}$  of type  $B_3, C_3$  or  $D_4$  containing this triple. Here  $\theta'$  is the maximal positive root in  $R'_+ = R' \cap \widetilde{R}_+$  unless for  $C_3$ ; in the latter case  $\theta'$  is the maximal short positive root.*

(ii) *Assuming that  $\widetilde{\alpha}$  satisfies (i), let  $\widetilde{\alpha}' \in \lambda(s_{\widetilde{\gamma}})$  be a root between  $\widetilde{\alpha}$  and  $\widetilde{\gamma}$  such that  $(\widetilde{\alpha}, \widetilde{\alpha}') = 0$  (see Lemma 1.2). Then a pair  $\{\widetilde{\alpha}', \widetilde{\alpha}\}$  can be made consecutive before  $\widetilde{\gamma}$  using rank two Coxeter transforms*



in  $\lambda(s_{\tilde{\gamma}})$ . These transforms are either in the form  $s_i s_j s_i \mapsto s_j s_i s_j$  with the midpoints corresponding to  $\tilde{\gamma}' = [\gamma, \nu_{\gamma} j'] \in \lambda(s_{\tilde{\gamma}})$  or in the form  $s_i s_j \mapsto s_j s_i$  otherwise.

(iii) Continuing,  $\lambda(s_{\tilde{\gamma}})$  can be transformed to a sequence with  $\tilde{\alpha}$  as the beginning and such that the only roots before  $\tilde{\gamma}$  non-orthogonal to  $\tilde{\alpha}$  in  $\lambda(s_{\tilde{\gamma}})$  are  $\tilde{\gamma}'$  from (ii) for  $j' < j$  and also  $\tilde{\beta}' = -s_{\tilde{\gamma}'}(\tilde{\alpha}) = \tilde{\gamma}' - \tilde{\alpha} \in \lambda(s_{\tilde{\gamma}})$ .

*Proof.* For reflections  $s_{\tilde{\gamma}}$ , the non-admissibility is exactly as in (i). Indeed, we do not need to check that  $\tilde{\alpha}$  is a unique simple root from  $R'_+ = R' \cap \tilde{R}_+$  in  $\lambda(s_{\tilde{\gamma}}) \cap R'$ , since all simple roots in  $R'$  but  $\tilde{\alpha}$  are orthogonal to such  $\tilde{\gamma}$ . For instance, the roots  $\epsilon_1 - \epsilon_2$  and  $m\epsilon_3$  are orthogonal to  $\tilde{\gamma} = \epsilon_1 + \epsilon_2$  and therefore cannot belong to  $\lambda(s_{\tilde{\gamma}})$  in the case of (2.48). Then claims (i,ii,iii) follow from the Main Theorem and Lemma 1.2). In (iii), we move  $\tilde{\alpha}$  to the position next to  $\tilde{\gamma}$ ; all roots it “passes” have to be orthogonal to  $\tilde{\alpha}$  unless they are in the form  $\tilde{\gamma}', \tilde{\beta}'$ . Then we can move  $\tilde{\alpha}$  back to its first position.  $\square$

#### 4. RIGHT BRUHAT ORDERING

We will define the Bruhat ordering on  $\widehat{W}$  for the system  $\tilde{R}$  relative to its root subsystem. The notations are from the previous sections with a reservation about  $\tilde{R}^0$ .

**4.1. Basic properties.** By a **root subsystem**  $\tilde{R}^0$ , we will mean the intersection of  $\tilde{R}$  with a  $\mathbb{Z}$ -lattice, namely,

$$\tilde{R}^0 = \tilde{R} \cap \tilde{\Lambda}^0, \quad \tilde{\Lambda}^0 \subset \tilde{Q} \stackrel{\text{def}}{=} \sum_{\tilde{\alpha} \in \tilde{R}} \mathbb{Z}\tilde{\alpha} = [Q, \mathbb{Z}];$$

it is obviously a reduced root system in its own right. *This meaning of  $\tilde{R}^0$  will be fixed till the end of the paper.* According to [B],  $\tilde{R}^0$  is a *subsystem* if and only if it is *symmetric* ( $-\tilde{R}^0 = \tilde{R}^0$ ) and *closed*, i.e.,

$$\tilde{\alpha} + \tilde{\beta} \in \tilde{R} \Rightarrow \tilde{\alpha} + \tilde{\beta} \in \tilde{R}^0 \quad \text{for } \tilde{\alpha}, \tilde{\beta} \in \tilde{R}^0.$$

For instance, all long roots always form a subsystem. We refer to [B] for the description of the maximal (proper) subsystems in the nonaffine case.

A subsystem may be reducible, then it is a direct sum of pairwise orthogonal irreducible reduced root systems. Almost all definitions and

constructions for irreducible systems can be extended to the reducible case; we use them for reducible systems without comments.

Note that *not* all root systems that belong to  $\tilde{R}$  are root subsystems in the above sense. For instance, in the notation from [B], the set  $B_2^{\text{sht}}$  of short roots in  $B_2$ ,

$$B_2^{\text{sht}} = \pm\{\alpha_1 + \alpha_2, \alpha_2\} \subset \pm\{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\},$$

does not correspond to any  $\Lambda^0$ . Indeed, otherwise it would contain  $\alpha_1$ , that is long.

The subset  $B_2^{\text{lng}} = \pm\{\alpha_1, \alpha_1 + 2\alpha_2\}$  of long roots in  $B_2$  is the intersection of  $B_2$  with the lattice  $\Lambda^0 = \{v \in Q, (v, \alpha_1) \in 4\mathbb{Z}\}$ ; the normalization is as above:  $(\alpha, \alpha) = 2$  for *short*  $\alpha$ .

It is also a simplest example of a *root subsystem* that is not an intersection of  $B_2$  with any linear  $\mathbb{Q}$ -subspace in the  $\mathbb{Q}$ -span of  $Q$ . Note that  $\tilde{R}^0$  is an intersection of  $\tilde{R}$  with a  $\mathbb{Q}$ -linear subspace in  $\mathbb{Q}^{n+1}$  if and only if  $\Lambda' \stackrel{\text{def}}{=} \tilde{Q}/\tilde{\Lambda}^0$  has no torsion.

The affine variant of this example is the set  $\tilde{B}_2^{\text{lng}}$  of *affine* long roots in  $[B_2, \mathbb{Z}]$ ; it is a root subsystem; associated with  $\tilde{\Lambda}^0 = [\Lambda^0, 2\mathbb{Z}]$ .

Thus, our *root subsystems* are closed with respect to the integral linear combinations whenever the results are roots, but  $\mathbb{Q}$ -linear combinations are not allowed here; *this class is between a wider class of all subsets that are root systems in their own right and a narrower class of the intersections with linear  $\mathbb{Q}$ -subspaces.*

**Relations between  $\tilde{R}$  and  $\tilde{R}^0$ .** We are going to supply  $\tilde{R}^0$  with the *induced* systems of positive and simple roots and establish connections of the  $\lambda$ -sets for the corresponding Weyl groups.

The subset of positive roots of  $\tilde{R}^0$  is defined as  $\tilde{R}_+^0 = \tilde{R}_+ \cap \tilde{R}^0$ ; its minimal elements form simple roots  $\{\alpha_j^0\}$  in  $\tilde{R}_+^0$  and  $\mathbb{Z}$ -generate the corresponding root lattice  $\tilde{Q}^0$ . Note that the intersection  $Q_+ \cap \tilde{Q}^0$  is generally greater than  $\tilde{Q}_+^0$  defined as the  $\mathbb{Z}_+$ -span of the simple roots in  $\tilde{R}_+^0$ ; an example is  $\tilde{R}^0 = B_2^{\text{lng}}$ . The coincidence holds, for instance, if every  $\alpha_{j'}^0$  contains a certain  $\alpha_{j'}$  for  $0 \leq j' \leq n$  in its decomposition such that distinct  $j$  have distinct  $j'$ .

An important property of *root subsystems* is the compatibility with taking the  $\lambda$ -sets of the elements from  $\tilde{W}^0$  defined for  $\tilde{R}^0$  and those for the main root system  $\tilde{R}$ . It is based on the fact that intersections

of  $\lambda$ -sets in  $\tilde{R}$  with  $\tilde{R}^0$  are  $\lambda$ -sets with respect to  $\tilde{R}_+^0 = \tilde{R}^0 \cap \tilde{R}_+$  and the corresponding simple roots in  $\tilde{R}_+^0$ . See Theorem 2.1.

**Proposition 4.1.** *Let  $\tilde{R}^0$  be a root subsystem of  $\tilde{R}$  with a natural induced subset  $\tilde{R}_+^0 = \tilde{R}_+ \cap \tilde{R}^0$  of positive roots and the corresponding set of simple roots in  $\tilde{R}_+^0$ ,  $\tilde{W}^0$  the Weyl group of  $\tilde{R}^0$  generated by the reflections  $s_{\tilde{\alpha}}$  for  $\tilde{\alpha} \in \tilde{R}^0$  (the simple reflections are sufficient),  $\lambda^0(\tilde{u})$  the  $\lambda$ -sets defined within  $\tilde{R}^0$  for  $\tilde{u} \in \tilde{W}^0$ .*

*Given  $\hat{w} \in \hat{W}$ , there exists a unique element  $\hat{w}|_0 \in \tilde{W}^0$  such that  $\lambda^0(\hat{w}|_0) = \lambda(\hat{w}) \cap \tilde{R}^0$ . Here the ordering of the roots is induced from that in  $\lambda(\hat{w})$ . Explicitly, setting  $g = |\lambda(\hat{w}) \cap \tilde{R}^0|$ ,*

$$(4.58) \quad \hat{w}|_0 = s^{p_1} \cdots s^{p_g}, \text{ where } \lambda(\hat{w}) \cap \tilde{R}^0 = \{\tilde{\alpha}^{p_g}, \dots, \tilde{\alpha}^{p_1}\}, s^p = s_{\tilde{\alpha}^p},$$

$$\hat{w}|_0 = \cdots (s^{p_1} s^{p_2} s^{p_3} s^{p_2} s^{p_1}) (s^{p_1} s^{p_2} s^{p_1}) (s^{p_1}) \text{ is reduced in } \tilde{W}^0,$$

*where the elements in  $(\cdot)$  are simple reflections in  $\tilde{W}^0$ .*

*Proof.* The conditions  $(a, b, c, d)$  from Theorem 2.1 are compatible with the intersections with  $\tilde{R}^0$ . Concerning  $(c, d)$ , we use the following: if the difference of two roots from  $\tilde{R}^0$  is a root in  $\tilde{R}$ , then it belongs to  $\tilde{R}^0$ . As for (4.58), formula (1.21) is applied.  $\square$

Let  $\tilde{R}^0 \subset \tilde{R}$  be a root subsystem of  $\tilde{R}$ . Given  $\hat{w} \in \hat{W}$  and its reduced decomposition  $\hat{w} = \pi_r s_{i_1} \cdots s_{i_l}$ , the **right Bruhat set**  $\mathcal{B}^0(\hat{w}) \subset \hat{W}$  with respect to  $\tilde{R}^0$  is formed by the products  $\hat{w}'$  obtained from the decomposition of  $\hat{w}$  by striking out any number of simple **right singular reflections**  $s_{i_p}$  satisfying by definition  $\tilde{\alpha}^p \in \tilde{R}^0$  in the notation  $\tilde{\alpha}^p = s_{i_1} \cdots s_{i_{p-1}}(\alpha_p)$  from the previous section (used in Proposition 4.1).

If the number of removed singular reflections is nonzero then the notation will be  $\mathcal{B}_o^0(\hat{w})$ ; thus  $\mathcal{B}^0(\hat{w}) = \mathcal{B}_o^0(\hat{w}) \cup \hat{w}$ .

**Comment.** The *left Bruhat ordering* with respect to  $\tilde{R}^0$  is the  $\hat{w} \mapsto \hat{w}^{-1}$  image of the right Bruhat ordering. Explicitly, it is defined by erasing (some of)  $s_{i_p}$  satisfying the conditions  $\pi_r s_{i_l} \cdots s_{i_{p+1}}(\alpha_p) \in \tilde{R}^0$  for reduced  $\hat{w} = \pi_r s_{i_l} \cdots s_{i_1}$ . Such simple reflections can be called *left singular*.

The left ordering is different from the right Bruhat ordering unless  $\tilde{R}^0$  is  $\hat{W}$ -invariant. They obviously coincide for the usual Bruhat ordering

on  $\widehat{W}$ , when  $\widetilde{R}^0 = \widetilde{R}$ . The example of the root subsystem  $\widetilde{B}_2^{\text{lng}}$  considered above satisfies the  $\widehat{W}$ -invariance condition and demonstrates that the coincidence of the right and left Bruhat orderings may occur for some non-trivial  $\widetilde{R}^0$ . Obviously,  $\widehat{W}$ -invariant nonzero  $\widetilde{R}^0$  must be of maximal rank due to the irreducibility of the action of  $\widehat{W}$  in  $\mathbb{R}^{n+1}$ . It is not difficult to describe all such cases, but we do not need it here.

*Only the right Bruhat ordering will be used in this paper; the word “right” will be mainly omitted.*  $\square$

**4.2. Using Coxeter transforms.** The next theorem shows that the properties of the right Bruhat ordering are similar to those of the usual one, with a reservation that it is not invariant with respect to the inversion  $\widehat{w} \mapsto \widehat{w}^{-1}$ .

Recall that the group generated by all  $s_{\widetilde{\alpha}}$  for  $\widetilde{\alpha} \in \widetilde{R}^0$  is denoted by  $\widetilde{W}^0$ . We put

$$(4.59) \quad \begin{aligned} \widehat{w} \geq_0 \widehat{w}' & \text{ if } \widehat{w}' \in \mathcal{B}^0(\widehat{w}), \text{ respectively,} \\ \widehat{w} >_0 \widehat{w}' & \text{ if } \widehat{w}' \in \mathcal{B}_o^0(\widehat{w}), \text{ i.e., if } \widehat{w}' \neq \widehat{w}. \end{aligned}$$

**Theorem 4.2.** *Given a root subsystem  $\widetilde{R}^0 \subset \widetilde{R}$  and a reduced decomposition  $\widehat{w} = \pi_r s_{i_1} \cdots s_{i_l}$  of  $\widehat{w} \in \widehat{W}$ ,*

(a)  $\mathcal{B}^0(\widehat{w})$  does not depend on the choice of the reduced decomposition of  $\widehat{w}$ ;

(b)  $\mathcal{B}^0(\widehat{w}') \subset \mathcal{B}^0(\widehat{w})$  if  $\widehat{w}' \in \mathcal{B}^0(\widehat{w})$ , i.e., if the ordering  $\geq_0$  is transitive;

(c)  $\mathcal{B}^0(\widehat{w}') \subset \mathcal{B}^0(\widehat{w}s_{\widetilde{\alpha}_p})$ , where  $s_{i_p}$  is the first (on the right) in the set of singular reflections deleted when constructing  $\widehat{w}' \in \mathcal{B}^0(\widehat{w})$ ;

(d)  $\cap_{\widehat{w}'} \mathcal{B}^0(\widehat{w}') = \{\widehat{w}^\circ\}$ , where  $\widehat{w}^\circ$  is obtained from  $\widehat{w}$  by crossing out all singular  $s_{i_p}$ ;

(e)  $\widehat{w}^\circ$  is a unique element of minimal length in the coset  $\widehat{w}\widetilde{W}^0$  for  $\widetilde{W}^0 = \langle s_{\widetilde{\alpha}} \mid \widetilde{\alpha} \in \widetilde{R}^0 \rangle$ ;

(f) if  $l(s_j \widehat{w}) < l(\widehat{w})$  for  $0 \leq j \leq n$  and also  $\widehat{w}^{-1}(\alpha_j) \in \widetilde{R}^0$ , then  $s_j \widehat{w} \in \mathcal{B}_o^0(\widehat{w})$ .

*Proof.* Setting  $s^p = s_{\widetilde{\alpha}^p}$ , the element  $\widehat{w}' \in \mathcal{B}(\widehat{w})$  obtained by striking out  $g > 0$  simple reflections  $s_{i_p}$  for the indices  $p$  from the sequence

$\{p_g >, \dots, > p_1\}$  equals  $\widehat{w}' = \widehat{w}s^{p_g} \dots s^{p_1}$ . This representation of  $\widehat{w}'$  will be used constantly.

We will extend the definition of  $\mathcal{B}^0(\widehat{w})$  to *possibly non-reduced*, decompositions of  $\widehat{w}$ . The notation  $\widetilde{\mathcal{B}}^0(\widehat{w})$  will be used; in this definition the *initial decomposition of  $\widehat{w}$  must be given*; this set may depend on its choice. The decompositions, of  $\widehat{w}'$ , possible non-reduced, that are obtained from a given decomposition of  $\widehat{w}$  by striking out (some of) the corresponding singular simple reflections will be called *standard*; the elements from  $\widetilde{\mathcal{B}}^0(\widehat{w})$  will be considered with the corresponding standard decompositions (unless stated otherwise). We will generally distinguish the elements from  $\widetilde{\mathcal{B}}^0(\widehat{w})$  with coinciding  $\widehat{w}'$  if their *standard decompositions* are different.

If the decomposition of  $\widehat{w}$  is fixed, then given a standard decomposition of  $\widehat{w}' \in \widetilde{\mathcal{B}}_o^0$ , its  $\widetilde{\lambda}$ -sequence, defined by formula (1.16), is

$$(4.60) \quad \widetilde{\lambda}(\widehat{w}') = \{(s^{p_1} \dots s^{p_g})(\widetilde{\alpha}^l), \dots, (s^{p_1} \dots s^{p_h})(\widetilde{\alpha}^q), \dots, \widetilde{\alpha}^1 = \alpha_{i_1}\},$$

where  $p_h$  is the last  $p$ -index such that  $s_{i_p}$  is removed before  $s_{i_q}$ . To be more exact, let  $q$  run through the set of all indices  $\{l, \dots, 1\}$  apart from the  $p$ -sequence, then  $p_h$  for  $h = h(q)$  in (4.60) is the greatest index  $p$  smaller than  $q$ .

We see that an arbitrary  $\widetilde{\beta} \in \widetilde{\lambda}(\widehat{w}')$  is naturally represented in the form  $\widetilde{\beta} = s^{p_1} \dots s^{p_h}(\widetilde{\alpha}^q)$ . Moreover,  $\widetilde{\beta}$  belongs to  $\widetilde{R}^0$  if and only if  $\widetilde{\alpha}^q \in \widetilde{R}^0$ ;  $\widetilde{\alpha}^q$  is defined for the initial decomposition of  $\widehat{w}$ . Indeed, the reflections  $s^p$  and their products preserve  $\widetilde{R}^0$  for all  $p \in \{p_g, \dots, p_1\}$  by construction.

We see that, given an initial decomposition of  $\widehat{w}$ , iterations (compositions) of the deleting procedure in the class of standard decompositions do not give anything new; deleting singular simple reflections in the standard decompositions (non-necessarily reduced) of  $\widehat{w}' \in \widetilde{\mathcal{B}}^0(\widehat{w})$  leads to  $\widehat{w}''$  (with the resulting standard decompositions) from the same set  $\widetilde{\mathcal{B}}^0(\widehat{w})$ .

Such transitivity is based on the fact that singular simple reflections *remain singular* in the corresponding  $\widehat{w}'$  due to (4.60).

The following lemma extends (a) from Theorem 4.2 to the class of non-reduced decompositions.

**Lemma 4.3.** *Homogeneous Coxeter transforms of a given, possibly non-reduced, decomposition of  $\widehat{w}$  do not change the set  $\widetilde{\mathcal{B}}^0(\widehat{w})$  considered simply as a set of elements in  $\widehat{W}$  (i.e., without the corresponding standard decompositions).*

*Proof.* From the view point of the sequence  $\widetilde{\lambda}(\widehat{w})$  from (1.16), given a Coxeter transform, the corresponding consecutive roots  $\widetilde{\alpha}^p$  constitute all positive roots of a root system of rank two; this transform will permute them changing their order in  $\widetilde{\lambda}(\widehat{w})$  to the inverse one.

The set  $\widetilde{\mathcal{B}}^0(\widehat{w})$  obvious remains unchanged if no singular  $\widetilde{\alpha}^p (\in \widetilde{R}^0)$  are involved in this permutation. Otherwise, the following configurations of simple singular reflections may occur.

*First*, only *one* singular  $\widetilde{\alpha}^p$  can be involved. We examine removing the corresponding simple singular reflection before and after the Coxeter transformation of type  $A_1 \times A_1$ ,  $A_2$ ,  $B_2$ , or  $G_2$ .

In the  $A_2$ -case, the singular element can be  $\underline{s_i s_{i+1} s_i}$ ,  $s_i \underline{s_{i+1} s_i}$ , or  $s_i s_{i+1} \underline{s_i}$ ; it is underlined. The transformation  $s_i s_{i+1} s_i \mapsto s_{i+1} s_i s_{i+1}$  in  $\widehat{w}$  becomes respectively

$$s_i s_{i+1} \mapsto s_i s_{i+1}, \quad s_{i+1} s_i \mapsto s_{i+1} s_i, \quad \text{or} \quad s_i^2 \mapsto s_{i+1}^2.$$

This is “inside”  $s_i s_{i+1} s_i$  in the decomposition of  $\widehat{w}$ . Obviously the corresponding three  $\widehat{w}'$  remain unchanged. The other rank two cases are equally simple.

*Second*, at least *two* singular simple reflections can be involved in the transformation. In the case of the Coxeter relation of type  $A_2$ , *all* simple reflections have to be singular since  $\widetilde{R}^0$  is stable under addition and subtraction. Therefore the right Bruhat sets for  $s_i s_{i+1} s_i$  and  $s_{i+1} s_i s_{i+1}$  are the usual Bruhat sets (the whole  $\mathbf{S}_3$  generated by  $s_i, s_{i+1}$  inside the decomposition of  $\widehat{w}$ ) and obviously coincide.

Similarly, the reference to the standard Bruhat ordering is sufficient in the remaining rank two cases,  $A_1 \times A_1$ ,  $B_2$  or  $G_2$  provided that *all* involved  $\widetilde{\alpha}$  are from  $\widetilde{R}^0$ .

*Third*, for  $B_2$ , the configuration  $\underline{s_i s_{i+1} s_i s_{i+1}}$  with exactly two singular reflections may occur; cf. the example of  $B_2^{\text{lng}}$  considered above. It becomes  $s_{i+1} \underline{s_i s_{i+1} s_i}$  upon the transformation

$$s_i s_{i+1} s_i s_{i+1} \mapsto s_{i+1} s_i s_{i+1} s_i.$$

The elements  $s_i$  (0, 1 or 2 of them) are allowed to be removed from the product  $s_i s_{i+1} s_i s_{i+1}$  and its inverse. The corresponding right Bruhat

sets (inside  $\widehat{w}$ ) obviously coincide:

$$\{s_i s_{i+1} s_i s_{i+1}, s_i s_{i+1}^2, s_{i+1} s_i s_{i+1}, \text{id}\}.$$

Here  $s_{i+1}$  may be underlined instead of  $\underline{s_i}$ ; it is completely analogous.

We can always treat the deleting procedure as right multiplication by proper reflections; then the corresponding elements  $\widehat{w}'$  are

$$\{\widehat{w}, \widehat{w}s_{\widetilde{\alpha}}, \widehat{w}s_{\widetilde{\beta}}, \widehat{w}s_{\widetilde{\alpha}}s_{\widetilde{\beta}}\},$$

where  $\widetilde{\alpha}, \widetilde{\beta}$  are orthogonal to each other. The compatibility with the  $B_2$ -transformation simply means that this 4-set remains unchanged if  $s_{\widetilde{\alpha}}$  and  $s_{\widetilde{\beta}}$  are transposed, which is obvious because these reflection commute due to the orthogonality of  $\widetilde{\alpha}, \widetilde{\beta}$ . The same argument works for  $G_2$  as exactly two simple reflections are singular.

The last, the *fourth*, case is  $G_2$  with three singular reflections. They come from the  $A_2$ -subsets of long or short roots in the  $G_2$ -system. It suffices to check that deleting 0, 1, 2 or all three reflections in the products  $s_{\widetilde{\alpha}}s_{\widetilde{\alpha}+\widetilde{\beta}}s_{\widetilde{\beta}}$  and, respectively, in its inverse  $s_{\widetilde{\beta}}s_{\widetilde{\alpha}+\widetilde{\beta}}s_{\widetilde{\alpha}}$  result in coinciding sets, provided that  $\widetilde{\alpha} + \widetilde{\beta}$  is a root. It formally follows from the  $A_2$ -consideration.

Directly, in terms of the simple reflections, we have either

$$\underline{s_1}s_2\underline{s_1}s_2\underline{s_1}s_2 \mapsto s_2\underline{s_1}s_2\underline{s_1}s_2\underline{s_1}$$

or the same products where  $s_2$  is underlined instead of  $s_1$ . Actually, the latter corresponds to a subset of all short roots in  $G_2$  and  $\mathbb{Z}$ -generates the whole system; so it is not a *root subsystem* in our sense and this case can be omitted. The underlined reflections (0, 1, 2 or all of them) are allowed to be removed from the products; the right Bruhat sets coincide.

Similarly, the configuration for  $G_2$  with two simple singular reflections (the third case) is

$$s_1s_2\underline{s_1}s_2\underline{s_1}s_2 \text{ or } \underline{s_1}s_2s_1\underline{s_2}s_1s_2.$$

It leads to another, direct, justification of the coincidence for such configuration.  $\square$

Lemma 4.3 obviously implies (a). Let us demonstrate that the transitivity from (b) is a straightforward corollary of this lemma too.

Generally, an arbitrary reduced decompositions of  $\widehat{w}' \in \mathcal{B}^0(\widehat{w})$  can be obtained from a standard decomposition, maybe non-reduced, of

$\widehat{w}'$  using homogeneous Coxeter transformations, removing the squares  $s_i^2$ , if any, then applying homogeneous Coxeter transformations again, removing  $s_i^2$  (if occur) and so on.

Respectively, the sequence  $\lambda(\widehat{w}')$  will be obtained from  $\widetilde{\lambda}(\widehat{w}')$  by inverting the order of roots in subsequences of type  $A_1 \times A_1$ ,  $A_2$ ,  $B_2$  or  $G_2$ , respectively with 2, 3, 4, 6 neighboring roots, and also by deleting neighboring pairs  $\{\widetilde{\alpha}, -\widetilde{\alpha}\} \subset \widetilde{\lambda}(\widehat{w}')$ , if any. Such deleting may diminish  $\widetilde{\mathcal{B}}^0(\widehat{w}')$  if  $s_i$  is singular; the former procedure does not change this set as has been already checked.

Thus  $\mathcal{B}^0(\widehat{w}') \subset \widetilde{\mathcal{B}}^0(\widehat{w}') \subset \mathcal{B}^0(\widehat{w})$ , which gives (b). Claim (c) results from (b) by induction with respect to the length  $l = l(\widehat{w})$ . Indeed, if  $s_{i_1}$  is removed when constructing  $\widehat{w}'$ , then  $\widehat{w}' \in \mathcal{B}^0(\pi_r s_{i_1} \cdots s_{i_2})$  because the decomposition  $\pi_r s_{i_1} \cdots s_{i_2}$  remains reduced. Otherwise,  $\widehat{w}' \in \mathcal{B}^0(\pi_r s_{i_1} \cdots s_{i_2})_{s_{i_1}}$ . In any case, we can proceed by induction.

Justification of (d) is similar to that of (b). First, deleting *all* singular simple reflections from a *standard* decomposition, possibly non-reduced, of an arbitrary  $\widehat{w}' \in \mathcal{B}^0(\widehat{w})$  results in the same element  $\widehat{w}^\circ$ ; recall that the latter is obtained from the initial reduced decomposition of  $\widehat{w}$  by deleting all singular simple reflections *at once*. Second, the construction of  $\widehat{w}^\circ$  is compatible with the homogeneous Coxeter relations, which can be checked following Lemma 4.3. Third, this construction is compatible with removing the squares  $s_i^2$  from the decompositions. It proves (d).

The characterization of  $\widehat{w}^\circ$  from (e) is verified as follows. The set  $\lambda(\widehat{w}^\circ)$  contains no roots from  $\widetilde{R}^0$ , because so does the set  $\widetilde{\lambda}(\widehat{w}^\circ)$  constructed for a standard decomposition. Let us check that this is a defining property of  $\widehat{w}^\circ$ .

**Lemma 4.4.** *There exists a unique element  $\widehat{w}^*$  in the coset  $\widehat{w}\widetilde{W}^0$  such that  $\lambda(\widehat{w}^*) \cap \widetilde{R}^0 = \emptyset$ ; it has minimal possible length in this coset and  $\widehat{w}^* = \widehat{w}^\circ$ .*

*Proof.* If there are two such elements  $\widehat{w}^\circ, \widehat{w}^*$ , then  $\widehat{w}^* = \widehat{w}^\circ \widehat{u}$  for  $\widehat{u} \in \widetilde{W}^0$ . One has  $\lambda(\widehat{w}^*) \subset \widehat{u}^{-1}(\lambda(\widehat{w}^\circ)) \cup \lambda(\widehat{u})$  and the former set is obtained from the latter by removing the pairs  $-\widetilde{\alpha}, \widetilde{\alpha}$ . If  $\widehat{u} \neq \text{id}$ , then  $\lambda(\widehat{u})$  contains at least one  $\widetilde{\alpha} \in \widetilde{R}^0$ . Such  $\widetilde{\alpha}$  must disappear in  $\lambda(\widehat{w}^*)$ , which is possible only if  $\widehat{u}^{-1}(\lambda(\widehat{w}^\circ))$  contains  $-\widetilde{\alpha}$ . However,  $\widehat{u}$  belongs to  $\widetilde{W}^0$  and  $\widehat{u}^{-1}(\lambda(\widehat{w}^\circ)) \cap \widetilde{R}^0 = \emptyset$ ; we come to a contradiction.



Elements  $\widehat{w}'$  of minimal length in  $\widetilde{W}^0$  cannot have  $\tilde{\alpha} \in \widetilde{R}^0$  in their  $\lambda$ -sets  $\lambda(\widehat{w}')$  because otherwise  $l(\widehat{w}'s_{\tilde{\alpha}}) < l(\widehat{w}')$  would hold. Therefore,  $\widehat{w}^\circ$  is of minimal length and unique with this property.

Note that the existence of a unique element of minimal length is essentially Corollary 3.4 from [Dy], well known for parabolic subgroups of Coxeter groups.  $\square$

Let us check (f). Given a reduced decomposition  $\widehat{w} = \pi_r s_{i_l} \cdots s_{i_1}$ ,

$$\alpha_j = \pi_r s_{i_l} \cdots s_{i_{p+1}}(\alpha_{i_p}) = \widehat{w} s_{i_1} \cdots s_{i_p}(\alpha_{i_p})$$

for certain  $p$ . Then the element  $s_j \widehat{w}$  is obtained from  $\widehat{w}$  by striking out  $s_{i_p}$ . Moreover,

$$\tilde{\alpha}^p = s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p}) = s_{i_1} \cdots s_{i_{p-1}} s_{i_p}(-\alpha_{i_p}) = -\widehat{w}^{-1}(\alpha_j) \in \widetilde{R}^0$$

and  $s_{i_p}$  has to be a singular simple reflection. It completes the proof of the theorem.  $\square$

**4.3. Bruhat ordering on  $\widetilde{W}^0$ .** The right Bruhat ordering can be reduced to the standard Bruhat ordering in  $\widetilde{W}^0$ , that is established in the next proposition. It readily gives another proof of the description of the minimal elements  $\{\widehat{w}^\circ\}$  from Lemma 4.4. This proposition gives a simpler approach to the construction of  $\widehat{w}^\circ$ , although the considerations based directly on the Coxeter relations have their own advantages, especially when the Bruhat construction (deleting simple reflections) is extended to Hecke algebras.

For instance, it is important to determine how the choice of a reduced decomposition of  $\widehat{w} \in \widehat{W}$  influences deleting (singular)  $T_i$  in the corresponding product  $T_{\widehat{w}}$ ; here one can follow Lemma 4.4. We note that *compatible pairs of  $R$ -matrices* to be discussed below are a certain formalization of this “direct” approach.

Given  $\widehat{w} = \pi_r s_{i_l} \cdots s_{i_1}$ , let  $p \in \{p_g > p_{g-1} > \cdots, p_j, \dots, > p_2 > p_1\}$  for the sequence of all singular  $1 \leq p_j \leq l$  as  $1 \leq j \leq g$ . We use the notation

$$(4.61) \quad \tilde{\beta}^j = \tilde{\alpha}^{i_p} = \widehat{w}^p(\alpha_{i_p}) \text{ for } p = p_j, \text{ where } \widehat{w}^p \stackrel{\text{def}}{=} s_{i_1} \cdots s_{i_{p-1}},$$

and also will need  $\widehat{w}_j$  obtained from  $\widehat{w}^{p_j}$  by deleting *all* singular simple reflections. We set  $\beta_j = \widehat{w}_j(\alpha_{p_j}) \in \widetilde{R}^0$ ,  $1 \leq j \leq g$ . Then (cf. (4.58))

$$\begin{aligned} \widehat{w}_1 &= \widehat{w}^1, \quad \widehat{w}_2 = s_{i_1} \cdots s_{i_{p_1-1}} s_{i_{p_1+1}} \cdots s_{i_{p_2-1}} = \widehat{w}^{p_2} s^{p_1}, \\ \dots, \quad \widehat{w}_g &= \widehat{w}^{p_g} s^{p_1} \cdots s^{p_{g-1}} \text{ using } s^p = s_{\widetilde{\alpha}^p}; \\ (4.62) \quad \beta_1 &= \widehat{w}_1(\alpha_{p_1}) = \widetilde{\beta}^1, \quad \beta_2 = \widehat{w}_2(\alpha_{p_2}), \dots, \beta_g = \widehat{w}_g(\alpha_{p_g}). \end{aligned}$$

**Proposition 4.5.** (i) *The minimal element  $\widehat{w}^\circ$  from  $(d, e)$  of Theorem 4.2 equals  $\widehat{w} \widehat{w}|_0^{-1}$  for the unique element  $\widehat{w}|_0 \in \widetilde{W}^0$  such that  $\lambda^0(\widehat{w}|_0) = \lambda(\widehat{w}) \cap \widetilde{R}^0$  constructed in (4.58).*

(ii) *Explicitly,  $\lambda^0(\widehat{w}|_0) = \{\widetilde{\beta}^j, j = 1, \dots, g\}$ ,*

$$(4.63) \quad \widehat{w}|_0 = s_{\beta_g} \cdots s_{\beta_1} = s_{\widetilde{\beta}^1} \cdots s_{\widetilde{\beta}^g},$$

*where all roots  $\beta_j$  are simple in  $\widetilde{R}_+^0 = \widetilde{R}^0 \cap \widetilde{R}_+$ , and the first product is a reduced decomposition of  $\widehat{w}|_0$  in terms of the simple reflections of  $\widetilde{W}^0$  defined with respect to  $\widetilde{R}_+^0$ .*

(iii) *The right Bruhat sets can be expressed in terms of the standard Bruhat sets  $\mathcal{B}(\widetilde{u}; \widetilde{R}_+^0) \subset \widetilde{W}^0$  for  $\widetilde{u} \in \widetilde{W}^0$  (defined for the root system  $\widetilde{R}_+^0$  and the corresponding set of simple roots there):*

$$\mathcal{B}^0(\widehat{w}) = \widehat{w}^\circ \mathcal{B}(\widehat{w}|_0; \widetilde{R}_+^0) \text{ for arbitrary } \widehat{w} \in \widehat{W}.$$

*Proof.* The formula for  $\widehat{w}^\circ$  in terms of  $\widehat{w}|_0$  readily follows from Proposition 4.1.

Given a reduced decomposition of  $\widehat{w}$ , the element  $\widehat{w}|_0$  is naturally represented as the product  $s_{\widetilde{\beta}^1} \cdots s_{\widetilde{\beta}^g}$  for  $\{\widetilde{\beta}^j\} = \widetilde{R}^0 \cap \lambda(\widehat{w})$  and coincides with  $s_{\beta_g} \cdots s_{\beta_1}$  in (4.63) by construction.

Recall that these two products correspond to moving simple singular reflections in the decomposition of  $\widehat{w}$  to the right beginning with the *last* singular reflection,  $s_{p_g}$ , and beginning with the *first* singular reflection, that is  $s_{p_1}$ , respectively. Notice that  $s_{\beta_j}$  appear in this product in the order inverse to the order of  $s_{\widetilde{\beta}^j}$ .

Using Theorem 2.1, we conclude that  $\{\widetilde{\beta}^j\}$  is a  $\lambda$ -set in  $\widetilde{R}_+^0$  and, therefore, it is exactly  $\lambda^0(\widehat{w}|_0)$ ; hence,  $\widehat{w}|_0 = s_{\beta_g} \cdots s_{\beta_1}$  is a reduced decomposition in terms of simple reflections. It completes (ii).

Any increasing sequence of *simple* reflection in the decomposition of  $\widehat{w}|_0$  is associated with a decreasing sequence of positive roots  $\{\widetilde{\alpha}^{p_j}\} \subset \widetilde{R}^0 \cap \lambda(\widehat{w})$  (the order becomes inverse) and, therefore, corresponds to

an increasing sequence of singular simple reflections in the initial decomposition of  $\widehat{w}$ . Deleting the former sequences of singular reflections corresponds to deleting the latter sequences, and the other way round. It gives (iii).  $\square$

## 5. DOUBLE HECKE ALGEBRAS

By  $m$ , we denote the least natural number such that  $(P, P) = (1/m)\mathbb{Z}$ . Thus  $m = 2$  for  $D_{2k}$ ,  $m = 1$  for  $B_{2k}$  and  $C_k$ , otherwise  $m = |\Pi|$ .

The double affine Hecke algebra depends on the parameters  $q, t_\nu, \nu \in \{\nu_\alpha\}$ . It will be defined over the ring

$$\mathbb{Q}_{q,t} \stackrel{\text{def}}{=} \mathbb{Q}[q^{\pm 1/m}, t^{\pm 1/2}]$$

formed by polynomials in terms of  $q^{\pm 1/m}$  and  $\{t_\nu^{\pm 1/2}\}$ . We set

$$(5.1) \quad \begin{aligned} t_{\tilde{\alpha}} &= t_\alpha = t_{\nu_\alpha}, \quad t_i = t_{\alpha_i}, \quad q_{\tilde{\alpha}} = q^{\nu_\alpha}, \quad q_i = q^{\nu_{\alpha_i}}, \\ \text{where } \tilde{\alpha} &= [\alpha, \nu_\alpha j] \in \widetilde{R}, \quad 0 \leq i \leq n. \end{aligned}$$

It will be convenient to use the parameters  $\{k_\nu\}$  together with  $\{t_\nu\}$ , setting

$$t_\alpha = t_\nu = q_\alpha^{k_\nu} \quad \text{for } \nu = \nu_\alpha, \quad \text{and } \rho_k = (1/2) \sum_{\alpha > 0} k_\alpha \alpha.$$

For instance, by  $q^{(\rho_k, \alpha)}$ , we mean  $\prod_{\nu \in \nu_R} t_\nu^{((\rho_\nu)^\vee, \alpha)}$ ; here  $\alpha \in R$ ,  $(\rho_\nu)^\vee = \rho_\nu / \nu$ , and this product contains only *integral* powers of  $t_{\text{sht}}$  and  $t_{\text{lng}}$ . Note that  $(\rho_k, \alpha_i^\vee) = k_i = k_{\alpha_i} = ((\rho_k)^\vee, \alpha_i)$  for  $i > 0$ ;  $(\rho_k)^\vee \stackrel{\text{def}}{=} \sum k_\nu (\rho_\nu)^\vee$ . Also,  $(\rho_k, b_+) = -(\rho_k, b_-)$  for  $b_+ = w_0(b_-)$  (see above).

For pairwise commutative  $X_1, \dots, X_n$ ,

$$(5.2) \quad \begin{aligned} X_{\tilde{b}} &= \prod_{i=1}^n X_i^{l_i} q^j \quad \text{if } \tilde{b} = [b, j], \quad \widehat{w}(X_{\tilde{b}}) = X_{\widehat{w}(\tilde{b})}. \\ \text{where } b &= \sum_{i=1}^n l_i \omega_i \in P, \quad j \in \frac{1}{m}\mathbb{Z}, \quad \widehat{w} \in \widehat{W}. \end{aligned}$$

For instance,  $X_0 \stackrel{\text{def}}{=} X_{\alpha_0} = qX_\vartheta^{-1}$ .

Later  $Y_{\tilde{b}} = Y_b q^{-j}$  will be needed. Note the negative sign of  $j$ . For instance,  $Y_0 \stackrel{\text{def}}{=} Y_{\alpha_0} = q^{-1}Y_\vartheta^{-1}$ .

We set  $(\tilde{b}, \tilde{c}) = (b, c)$  ignoring the affine extensions.

**5.1. Main Definition.** We will use that  $\pi_r^{-1}$  is  $\pi_{r^*}$  and  $u_r^{-1}$  is  $u_{r^*}$  for  $r^* \in O$ ,  $u_r = \pi_r^{-1}\omega_r$ . The reflection  $*$  is induced by an involution of the nonaffine Dynkin diagram  $\Gamma$ .

**Definition 5.1.** *The double affine Hecke algebra  $\mathcal{H}$  is generated over  $\mathbb{Q}_{q,t}$  by the elements  $\{T_i, 0 \leq i \leq n\}$ , pairwise commutative  $\{X_b, b \in P\}$  satisfying (5.2), and the group  $\Pi$ , where the following relations are imposed:*

- (o)  $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0, 0 \leq i \leq n;$
- (i)  $T_i T_j T_i \dots = T_j T_i T_j \dots, m_{ij} \text{ factors on each side};$
- (ii)  $\pi_r T_i \pi_r^{-1} = T_j \text{ if } \pi_r(\alpha_i) = \alpha_j;$
- (iii)  $T_i X_b T_i = X_b X_{\alpha_i}^{-1} \text{ if } (b, \alpha_i^\vee) = 1, 0 \leq i \leq n;$
- (iv)  $T_i X_b = X_b T_i \text{ if } (b, \alpha_i^\vee) = 0 \text{ for } 0 \leq i \leq n;$
- (v)  $\pi_r X_b \pi_r^{-1} = X_{\pi_r(b)} = X_{u_r^{-1}(b)} q^{(\omega_{r^*}, b)}, r \in O'.$

□

Given  $\tilde{w} \in \widetilde{W}, r \in O$ , the product

$$(5.3) \quad T_{\pi_r \tilde{w}} \stackrel{\text{def}}{=} \pi_r \prod_{k=1}^l T_{i_k}, \text{ where } \tilde{w} = \prod_{k=1}^l s_{i_k}, l = l(\tilde{w}),$$

does not depend on the choice of the reduced decomposition (because  $\{T\}$  satisfy the same “braid” relations as  $\{s\}$  do). Moreover,

$$(5.4) \quad T_{\hat{v}} T_{\hat{w}} = T_{\hat{v}\hat{w}} \text{ whenever } l(\hat{v}\hat{w}) = l(\hat{v}) + l(\hat{w}) \text{ for } \hat{v}, \hat{w} \in \widehat{W}.$$

In particular, we arrive at the pairwise commutative elements

$$(5.5) \quad Y_b = \prod_{i=1}^n Y_i^{l_i} \text{ if } b = \sum_{i=1}^n l_i \omega_i \in P, \text{ where } Y_i \stackrel{\text{def}}{=} T_{\omega_i},$$

satisfying the relations

$$(5.6) \quad \begin{aligned} T_i^{-1} Y_b T_i^{-1} &= Y_b Y_{\alpha_i}^{-1} \text{ if } (b, \alpha_i^\vee) = 1, \\ T_i Y_b &= Y_b T_i \text{ if } (b, \alpha_i^\vee) = 0, 1 \leq i \leq n. \end{aligned}$$

The origin of this construction is due to Bernstein and Zelevinsky. The definition (5.5) and relations (5.6) are due to Lusztig (see, e.g., [L]); in the DAHA context, they are from [C3].

**5.2. Automorphisms.** The following maps can be uniquely extended to automorphisms of  $\mathcal{H}$  (see [C3],[C6],[C9]):

$$(5.7) \quad \varepsilon : X_i \mapsto Y_i, Y_i \mapsto X_i, T_i \mapsto T_i^{-1} (i \geq 1), t_\nu \mapsto t_\nu^{-1}, q \mapsto q^{-1},$$

$$\tau_+ : X_b \mapsto X_b, Y_r \mapsto X_r Y_r q^{-\frac{(\omega_r, \omega_r)}{2}}, \pi_r \mapsto q^{-(\omega_r, \omega_r)} X_r \pi_r,$$

$$(5.8) \quad \tau_+ : Y_\vartheta \mapsto q^{-1} X_\vartheta T_0^{-1} T_{s_\vartheta}, T_0 \mapsto q^{-1} X_\vartheta T_0^{-1}, \text{ and}$$

$$(5.9) \quad \tau_- \stackrel{\text{def}}{=} \varepsilon \tau_+ \varepsilon, \sigma \stackrel{\text{def}}{=} \tau_+ \tau_-^{-1} \tau_+ = \tau_-^{-1} \tau_+ \tau_-^{-1} = \varepsilon \sigma^{-1} \varepsilon,$$

where  $r \in O'$ . They fix  $T_i (i \geq 1)$ . The  $\tau_\pm, \sigma$  fix  $t_\nu, q$  and fractional powers of  $t_\nu, q$ .

Note that  $\tau_-$  acts trivially on  $\{T_0, \pi_r, Y_b\}$ ,  $\sigma$  sends  $X_b$  to  $Y_b^{-1}$ .

In the definition of  $\tau_\pm$  and  $\sigma$ , we need to add  $q^{\pm 1/(2m)}$  to  $\mathbb{Q}_{q,t}$ . These automorphisms actually act in the central extension of the *elliptic braid group* defined by the relations of  $\mathcal{H}$  where the quadratic relation is dropped and fractional powers of  $q$  are treated as central elements.

The elements  $\tau_\pm$  generate the projective  $PSL(2, \mathbb{Z})$ , which is isomorphic to the braid group  $B_3$  due to Steinberg. Adding  $\varepsilon$ , we obtain a projective action of  $PGL(2, \mathbb{Z})$ .

The following *anti-involutions* are of key importance for the theory of the polynomial representation:

$$(5.10) \quad X^* = X^{-1}, Y^* = Y^{-1}, T_i^* = T_i^{-1}, q^* = q^{-1}, t_\nu^* = t_\nu^{-1},$$

$$(5.11) \quad \phi \stackrel{\text{def}}{=} \varepsilon \star = \star \varepsilon : X_b \mapsto Y_b^{-1}, T_i \mapsto T_i (1 \leq i \leq n),$$

where the latter preserves  $q, t_\nu$  and their fractional powers.

We will also use  $\eta \stackrel{\text{def}}{=} \varepsilon \sigma$ . It conjugates  $q, t$  and is uniquely defined from the relations

$$\eta : T_i \mapsto T_i^{-1}, X_b \mapsto X_b^{-1}, \pi_r \mapsto \pi_r,$$

$$(5.12) \quad \text{where } 0 \leq i \leq n, b \in P, r \in O'.$$

**5.3. Intertwining operators.** The  $X$ -intertwiners (see [C7]) are introduced as follows:

$$(5.13) \quad \begin{aligned} \Phi_i &= T_i + (t_i^{1/2} - t_i^{-1/2})(X_{\alpha_i} - 1)^{-1} \text{ for } 0 \leq i \leq n, \\ \Phi_i^\diamond &= \Phi_i(X_{\alpha_i} - 1) = T_i(X_{\alpha_i} - 1) + t_i^{1/2} - t_i^{-1/2}, \\ G_i &= \Phi_i(\phi_i)^{-1}, \phi_i = t_i^{1/2} + (t_i^{1/2} - t_i^{-1/2})(X_{\alpha_i} - 1)^{-1}. \end{aligned}$$

They belong to  $\mathcal{H}$  extended by the rational functions in terms of  $\{X\}$ . The  $G$  are called the **normalized intertwiners**. The elements  $G_i, 0 \leq i \leq n, r \in O'$ , satisfy the same relations as  $\{s_i, \pi_r\}$  do, so the map

$$(5.14) \quad \widehat{w} \mapsto G_{\widehat{w}} = \pi_r G_{i_l} \cdots G_{i_1}, \quad \text{where } \widehat{w} = \pi_r s_{i_l} \cdots s_{i_1} \in \widehat{W},$$

is a well defined homomorphism from  $\widehat{W}$ .

The intertwining property is

$$G_{\widehat{w}} X_b G_{\widehat{w}}^{-1} = X_{\widehat{w}(b)} \quad \text{where } X_{[b,j]} = X_b q^j.$$

We will refer to  $G_{\widehat{w}}$  as intertwiners too;  $\Phi_i, \Phi_i^\diamond, G_i, \pi_r$  will be called *simple intertwiners*.

As to  $\Phi_i$  and  $\Phi_i^\diamond$ , they satisfy the homogeneous Coxeter relations and those with  $\Pi_r$ . So we may set  $\Phi_{\widehat{w}} = \pi_r \Phi_{i_l} \cdots \Phi_{i_1}$  for the reduced decompositions; similarly,

$$\Phi_{\widehat{w}}^\diamond = \pi_r \Phi_{i_l}^\diamond \cdots \Phi_{i_1}^\diamond = \Phi_{\widehat{w}}(X_{\alpha^l} - 1) \cdots (X_{\alpha^1} - 1)$$

in the notation from (1.16):  $\widetilde{\alpha}^p = s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p})$ . They intertwine  $X$  in the same way as  $G$  do.

The  $\Phi_w$  for  $w \in W$  are well known in the theory of affine Hecke algebras. The affine intertwiners are the key tool in the theory of semisimple and spherical representations of DAHA, including applications to the Macdonald polynomials and the Harish-Chandra–Opdam spherical transform.

**5.4. Intertwiners and  $T_i$ .** Concerning the relation of  $\Phi$  to  $\{T_i\}$ , the following holds:

$$(5.15) \quad T_i \Phi_{\widehat{w}} = \Phi_{\widehat{w}} T_j \quad \text{if } s_i \widehat{w} = \widehat{w} s_j, \quad l(\widehat{w} s_j) = l(\widehat{w}) + 1 = l(s_i \widehat{w})$$

for  $i, j \geq 0$ . These conditions are equivalent to  $s_j(\lambda(\widehat{w})) = \lambda(\widehat{w})$  and  $\widehat{w}(\alpha_j) = \alpha_i$ .

An explicit description of such  $\widehat{w}$  and the justification are straightforward. Indeed, there must exist a reduced decomposition of  $\widehat{w}$  that begins with  $\widehat{u}$  such that  $\widehat{u} s_j = s_{j'} \widehat{u}$  is an elementary Coxeter transformation ( $j' = j$  unless it is of type  $A_2$ ). One has:

$$(5.16) \quad \begin{aligned} T_{j'} \Phi_{\widehat{u}} &= (\Phi_{j'} - (t_{j'}^{1/2} - t_{j'}^{-1/2})/(X_{\alpha_{j'}} - 1)) \Phi_{\widehat{u}} \\ &= \Phi_{\widehat{u}} \Phi_{j'} - \Phi_{\widehat{u}} (t_j^{1/2} - t_j^{-1/2})/(X_{\alpha_j} - 1) = \Phi_{\widehat{u}} T_j. \end{aligned}$$

Then  $s_i \widehat{w}' = \widehat{w}' s_{j'}$  for  $\widehat{w} = \widehat{w}' \widehat{u}$  and we can continue by induction.

Let us generalize (5.15) using the right Bruhat ordering from Section 4 associated with an arbitrary root subsystem  $\tilde{R}^0$  of  $\tilde{R}$ .

**Theorem 5.2.** *Given a reduced decomposition  $\hat{w} = \pi_r s_{i_1} \cdots s_{i_1}$ , we define  $\tilde{\Phi}_{\hat{w}}$  by replacing  $\Phi_{i_p}$  with  $T_{i_p}$  when  $p$  is **singular**, i.e.,  $\tilde{\alpha}^p = s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p}) \in \tilde{R}^0$ ; the  $\tilde{\Phi}_{\hat{w}}^\circ$  and  $\tilde{G}_{\hat{w}}$  are defined correspondingly.*

(a) *The element  $\tilde{\Phi}_{\hat{w}}$  remains unchanged under homogeneous Coxeter transformations of the reduced decomposition of  $\hat{w}$  if the  $A_2$ -transforms with singular middle  $i_p$  and non-singular edges are avoided, that are as follows*

$$(5.17) \quad s_{i \pm 1} s_i s_{i \pm 1} \mapsto s_i s_{i \pm 1} s_i, \quad \text{for } i = i_p, \quad i_{p+1} = i \pm 1 = i_{p-1},$$

where  $\tilde{\alpha}^p \in \tilde{R}^0$  and  $\tilde{\alpha}^{p-1} \notin \tilde{R}^0 \not\supset \tilde{\alpha}^{p+1}$ .

(b) *Similarly,  $\tilde{\Phi}_{\hat{w}}^\circ$  are invariant under homogeneous Coxeter transformations apart from (5.17). This restriction is not needed for  $\tilde{G}_{\hat{w}}$ , i.e., the latter does not depend on the choice of the reduced decomposition for arbitrary homogeneous Coxeter transformations if the normalized intertwiners are used.*

(c) *If  $\tilde{\Phi}^2$  is obtained from  $\tilde{\Phi}^1 = \tilde{\Phi}_{\hat{w}}$  when transformations (5.17) are allowed, then the difference  $\tilde{\Phi}^1 - \tilde{\Phi}^2$  is a linear combination of the terms in the form  $\tilde{\Phi}_{\hat{w}'} P(X)$  for rational functions  $P(X)$  from*

$$(5.18) \quad \mathbb{Z}[q_\nu, (t_\nu^{1/2} - t_\nu^{-1/2}), X_\beta, (1 - X_{\tilde{\alpha}})^{-1}], \quad \beta \in R, \tilde{\alpha} \in \tilde{R} \setminus \tilde{R}^0,$$

where  $\tilde{\Phi}_{\hat{w}'}$  are defined for some reduced decompositions of  $\hat{w}' \in \mathcal{B}_o^0(\hat{w})$  from Theorem 4.2.

(d) *The linear space generated by  $\tilde{\Phi}_{\hat{w}'} P(X)$  for  $\hat{w}' \in \mathcal{B}^0(\hat{w})$  and*

$$P(X) \in \mathbb{Q}_{q,t}[X_b, (1 - X_{\tilde{\alpha}})^{-1} \mid b \in B, \tilde{\alpha} \notin \tilde{R}^0]$$

*is a left module over  $\mathbb{Q}_{q,t}[X_b, (1 - X_{\tilde{\alpha}})^{-1} \mid b \in B, \tilde{\alpha} \notin \hat{w}^{-1}(\tilde{R}^0)]$ .*

*Proof.* All Coxeter rank 2 transformations but the exceptional one from (5.17) are checked similar to (5.16); one follows the proof of Theorem 4.2.

The cases of  $\tilde{\Phi}^\circ$  and  $\tilde{G}$  are completely analogous. The unitary property  $G_i^2 = 1$  guarantees that the transformations (5.17) do not change  $\tilde{G}_{\hat{w}}$ . We arrive at (a, b).

Let the transformation be  $s_{i \pm 1} \underline{s_i} s_{i \pm 1} \mapsto s_i \underline{s_{i \pm 1}} s_i$ , where we underline the singular reflection and  $i = i_p$ ,  $i_{p-1} = i \pm 1 = i_{p+1}$  inside a

decomposition of  $\widehat{w}$ . Then, setting  $\alpha = \alpha_i$ ,  $\beta = \alpha_{i\pm 1}$ ,  $t = t_i$ ,

$$(5.19) \quad \begin{aligned} \widetilde{\Phi}^1 &= \Phi_{i\pm 1} T_i \Phi_{i\pm 1}, \quad \widetilde{\Phi}^2 = \Phi_i T_{i\pm 1} \Phi_i, \\ \widetilde{\Phi}^1 - \widetilde{\Phi}^2 &= \frac{(t^{1/2} - t^{-1/2})^3 (X_\beta^{-1} - X_\alpha^{-1})}{(2 - X_\alpha - X_\alpha^{-1})(2 - X_\beta - X_\beta^{-1})}. \end{aligned}$$

The denominators of (5.19) become products of  $(X_{\tilde{\alpha}} - 1)$  for  $\tilde{\alpha} \notin \widetilde{R}^0$  when moved to the right all the way. We move them through non-singular  $\Phi_j$  using the intertwining property and through singular  $T_j$  using directly the relations (6.1). In either case, the denominators are conjugated by the corresponding  $s_j$ .

The product  $\widetilde{\Phi}^1$  where  $\Phi_{i\pm 1} T_i \Phi_{i\pm 1}$  is omitted becomes  $\widetilde{\Phi}_{\widehat{w}'}$  defined for the *standard*, possibly non-reduced, decomposition of  $\widehat{w}'$  obtained from  $\widehat{w}$  by deleting  $s_{i\pm 1} s_i s_{i\pm 1}$ . If this decomposition is non-reduced, then one needs either the quadratic relations for  $T_j^2$  for singular  $s_j$  or the formula for  $\Phi_j^2$ , an  $X$ -function, for non-singular  $s_j$ . Using these relations may be preceded by homogeneous Coxeter transformations. It results in additional terms of the same type (5.18). Eventually, we will come to  $\Phi_{\widehat{w}'}$  for reduced decompositions of  $\widehat{w}' \in \mathcal{B}^0(\widehat{w})$ , as required in (c).

Another transformation that may occur here is “deleting” singular  $T_j$  due to (6.1) in process of moving  $X$ -functions to the right, maybe followed by further reductions described above. The  $X$ -function here will be either  $\Phi_j^2$  or those due to (6.1); more generally, one can take here arbitrary rational functions of  $X$  with the denominators that become from (5.18) when moved to the right. This proves (c) and (d).  $\square$

Note that when  $(X_{\alpha_i} - X_{\alpha_{i\pm 1}})$  is moved to the right it becomes divisible by  $(X_{\tilde{\alpha}^p} - 1)$  for  $i = i_p$  in the absence of  $\{T_j\}$  in the corresponding portion of  $\widetilde{\Phi}_{\widehat{w}}$ . Otherwise, we will need to move this difference through  $T_j$  for singular indices  $j = i_q$  ( $q < p$ ), which may destroy the divisibility. More generally, we obtain the following.

**Corollary 5.3.** *Given a reduced decomposition  $\widehat{w} = \pi_r s_l \cdots s_1$  and a singular  $s_{i_p}$ , if  $s^p s^q = s^q s^p$  for all singular  $i_q$  with  $q < p$ , where the notation  $s^q = s_{\tilde{\alpha}^q}$  from Theorem 4.2 is used, then  $\widetilde{\Phi}^1 - \widetilde{\Phi}^2$  is divisible by  $(X_{\tilde{\alpha}^p} - 1)$  on the right.*  $\square$



**5.5. Compatible  $\mathcal{R}$ -matrices.** Generalizing the construction of  $\tilde{\Phi}$ , we come to the notion of the compatibility of  $\mathcal{R}$ -matrices. It extends the key property of the (quantum)  $\mathcal{R}$ -matrix  $\{\mathcal{R}_{\tilde{\alpha}}, \tilde{\alpha} \in \tilde{R}_+\}$  defined in [C2], which is as follows. The products  $\mathcal{R}_{\tilde{w}} = \mathcal{R}_{\tilde{\alpha}^l} \cdots \mathcal{R}_{\tilde{\alpha}^1}$  must depend only on  $\tilde{w}$  and must not depend on the *reduced* decompositions  $\tilde{w} = s_{i_l} \cdots s_{i_1}$  for any  $\tilde{w} \in \tilde{W}$ . Here  $\mathcal{R}_{\tilde{\alpha}}$  are elements in some algebra  $\mathfrak{R}$  with a unit. Equivalently, the *cocycle relation* must be satisfied:

$$(5.20) \quad \mathcal{R}_{\tilde{w}} \mathcal{R}_{\tilde{u}} = \tilde{u}^{-1}(\mathcal{R}_{\tilde{w}}) \mathcal{R}_{\tilde{u}}, \text{ where } \tilde{u}^{-1}(\dots \mathcal{R}_{\tilde{\alpha}} \dots) = \dots \mathcal{R}_{\tilde{u}^{-1}(\tilde{\alpha})} \dots,$$

provided that  $l(\tilde{u}\tilde{w}) = l(\tilde{u}) + l(\tilde{w})$ .

If  $\{\mathcal{R}_{\tilde{w}}, \tilde{w} \in \tilde{W}\}$  is extended to  $\widehat{W} \ni \hat{w} = \pi_r \tilde{w}$  by the formula  $\mathcal{R}_{\hat{w}} = \mathcal{P}_r \mathcal{R}_{\tilde{w}}$  for a homomorphism  $\Pi \ni \pi_r \mapsto \mathcal{P}_r \in \mathfrak{R}$  and (5.20) holds, then  $\mathcal{R}_{\hat{w}}$  is called a  $\widehat{W}$ -*extension* (or a  $\widehat{W}^b$ -extension for  $\Pi^b$  instead of  $\Pi$ ). It simply means the commutativity relations:  $\mathcal{P}_r \mathcal{R}_{\tilde{\alpha}} = \mathcal{R}_{\tilde{\alpha}} \mathcal{P}_r$ .

Given a *root subsystem*  $\tilde{R}^0$ , an  $\mathcal{R}$ -matrix  $\{\mathcal{R}_{\tilde{\alpha}}^0\}$  defined for  $\tilde{\alpha} \in \tilde{R}_+^0$  is called **completely compatible** with  $\mathcal{R}$  if the following *compatibility property* holds. The elements  $\tilde{\mathcal{R}}_{\tilde{w}}$  obtained from  $\mathcal{R}_{\tilde{w}}$  by replacing  $\mathcal{R}_{\tilde{\alpha}} \mapsto \mathcal{R}_{\tilde{\alpha}}^0$  whenever  $\tilde{\alpha} \in \tilde{R}_+^0$  have to remain unchanged when the (homogeneous) Coxeter transformations are applied to reduced decompositions of  $\tilde{w}$ , i.e., they must depend only on  $\tilde{w}$ . Note that the complete compatibility does not result in (5.20) for  $\tilde{\mathcal{R}}_{\tilde{w}}$  unless  $\tilde{R}^0$  is  $\widehat{W}$ -invariant.

Respectively, if  $\mathcal{P}_r \tilde{\mathcal{R}}_{\tilde{\alpha}} = \tilde{\mathcal{R}}_{\tilde{\alpha}} \mathcal{P}_r$ , then  $\{\tilde{\mathcal{R}}_{\hat{w}}, \hat{w} \in \widehat{W}\}$  is called a  $\widehat{W}$ -*extension* of  $\{\tilde{\mathcal{R}}_{\tilde{w}}, \tilde{w} \in \tilde{W}\}$ .

In the context of  $\mathcal{R}$ -matrices, the Coxeter transformations are *inversions* ( $w_0$ -permutations) of the consecutive elements  $\mathcal{R}_{\tilde{\alpha}^p}$  for the segments  $\{\tilde{\alpha}^p\} \subset \lambda(\tilde{w})$  that can be identified with consecutive positive roots of rank 2, i.e., with those of types  $A_1 \times A_1$ ,  $A_2$ ,  $B_2$ , or  $G_2$ .

An  $\mathcal{R}$ -matrix  $\mathcal{R}^0$  is called **partially compatible** with  $\mathcal{R}$  if the transformations from (5.17) are excluded, that are transformations of type  $A_2$  such that the second (middle) root belongs to  $\tilde{R}^0$  but the first and the third (the edges) do not.

**Rank two relations.** First of all, all rank two relations from [C2] for  $\mathcal{R}$  and  $\tilde{\mathcal{R}}$  (individually) must be satisfied. Then one needs to check the following rank two *compatibility relations*:

$$(A_1 \times A_1) : \tilde{\mathcal{R}}_{\alpha} \mathcal{R}_{\beta} = \mathcal{R}_{\beta} \tilde{\mathcal{R}}_{\alpha},$$

$$\begin{aligned}
& (A_2, \text{ general}) : \tilde{\mathcal{R}}_\alpha \mathcal{R}_{\alpha+\beta} \mathcal{R}_\beta = \mathcal{R}_\beta \mathcal{R}_{\alpha+\beta} \tilde{\mathcal{R}}_\alpha, \\
(5.21) \quad & (A_2, \text{ special}) : \mathcal{R}_\alpha \tilde{\mathcal{R}}_{\alpha+\beta} \mathcal{R}_\beta = \mathcal{R}_\beta \tilde{\mathcal{R}}_{\alpha+\beta} \mathcal{R}_\alpha,
\end{aligned}$$

where the *special relation* is needed to ensure the *complete compatibility* of  $\mathcal{R}$  and  $\tilde{\mathcal{R}}$  (otherwise, the compatibility will be only *partial*), and

$$\begin{aligned}
(B_2, 1 \sim) : & \tilde{\mathcal{R}}_\alpha \mathcal{R}_{\alpha+\beta} \mathcal{R}_{\alpha+2\beta} \mathcal{R}_\beta = \mathcal{R}_\beta \mathcal{R}_{\alpha+2\beta} \mathcal{R}_{\alpha+\beta} \tilde{\mathcal{R}}_\alpha \\
& \mathcal{R}_\alpha \tilde{\mathcal{R}}_{\alpha+\beta} \mathcal{R}_{\alpha+2\beta} \mathcal{R}_\beta = \mathcal{R}_\beta \mathcal{R}_{\alpha+2\beta} \tilde{\mathcal{R}}_{\alpha+\beta} \mathcal{R}_\alpha \\
& \mathcal{R}_\alpha \mathcal{R}_{\alpha+\beta} \tilde{\mathcal{R}}_{\alpha+2\beta} \mathcal{R}_\beta = \mathcal{R}_\beta \tilde{\mathcal{R}}_{\alpha+2\beta} \mathcal{R}_{\alpha+\beta} \mathcal{R}_\alpha \\
& \mathcal{R}_\alpha \mathcal{R}_{\alpha+\beta} \mathcal{R}_{\alpha+2\beta} \tilde{\mathcal{R}}_\beta = \tilde{\mathcal{R}}_\beta \mathcal{R}_{\alpha+2\beta} \mathcal{R}_{\alpha+\beta} \mathcal{R}_\alpha, \\
(B_2, 2 \sim) : & \tilde{\mathcal{R}}_\alpha \mathcal{R}_{\alpha+\beta} \tilde{\mathcal{R}}_{\alpha+2\beta} \mathcal{R}_\beta = \mathcal{R}_\beta \tilde{\mathcal{R}}_{\alpha+2\beta} \mathcal{R}_{\alpha+\beta} \tilde{\mathcal{R}}_\alpha \\
& \mathcal{R}_\alpha \tilde{\mathcal{R}}_{\alpha+\beta} \mathcal{R}_{\alpha+2\beta} \tilde{\mathcal{R}}_\beta = \tilde{\mathcal{R}}_\beta \mathcal{R}_{\alpha+2\beta} \tilde{\mathcal{R}}_{\alpha+\beta} \mathcal{R}_\alpha,
\end{aligned}$$

and also the corresponding compatibility relations for  $G_2$ , obtained from the ones for  $\mathcal{R}$ ,

$$\mathcal{R}_\alpha \mathcal{R}_{\alpha+\beta} \mathcal{R}_{2\alpha+3\beta} \mathcal{R}_{\alpha+2\beta} \mathcal{R}_{\alpha+3\beta} \mathcal{R}_\beta = \mathcal{R}_\beta \mathcal{R}_{\alpha+3\beta} \mathcal{R}_{\alpha+2\beta} \mathcal{R}_{2\alpha+3\beta} \mathcal{R}_{\alpha+\beta} \mathcal{R}_\alpha,$$

by replacing  $\mathcal{R} \mapsto \tilde{\mathcal{R}}$  for: 1) *one* of the roots ( $1 \sim$ ) from the set

$$\alpha, \alpha + \beta, 2\alpha + 3\beta, \alpha + 2\beta, \alpha + 3\beta, \beta,$$

2) those in the pairs  $\{\alpha, \alpha + 2\beta\}$   $\{2\alpha + 3\beta, \beta\}$  of orthogonal roots ( $2 \sim$ ) and 3) for those in the  $A_2$ -subsystems ( $3 \sim$ ):

$$\{\alpha, 2\alpha + 3\beta, \alpha + 3\beta\} \text{ or } \{\alpha + \beta, \alpha + 2\beta, \beta\}.$$

Here the roots involved constitute all positive roots in the corresponding rank two subsystem taken clockwise and then counterclockwise. See the proof of Lemma 4.3. The roots  $\alpha, \beta$  generating a rank two system are *affine* from  $\tilde{R}$ .

We mention that not all compatibility relations from these lists can appear for a given pair  $\tilde{R}^0 \subset R$  due to our definition of a *root subsystem*. The relations we give are “generic”; not all of them must be checked for concrete  $\tilde{R}^0 \subset R$ .

For instance, the subsets of *all* short roots do not form a *root subsystem* for  $B_2$  or  $G_2$ , as well as the pairs of orthogonal roots for  $G_2$ ; such roots  $\mathbb{Z}$ -generate the whole rank two system. The corresponding compatibility  $\mathcal{R}$ -relations have to be omitted if we stick to the definition of the *root subsystem*  $\tilde{R}^0$  from this paper.

**5.6. Examples.** Let us assume that there exists a homomorphism  $\widetilde{W} \rightarrow \mathfrak{R}^*$  sending  $s_{\tilde{\alpha}} \mapsto \mathcal{S}_{\tilde{\alpha}}$  and  $\mathcal{S}_{\tilde{\alpha}}(\mathcal{R}_{\tilde{\beta}})\mathcal{S}_{\tilde{\alpha}} = \mathcal{R}_{s_{\tilde{\alpha}}(\tilde{\beta})}$ . The simplest abstract example of *complete* (respectively, *partial*) compatibility is  $\{\mathcal{R}_{\tilde{\alpha}}^0 = \mathcal{S}_{\tilde{\alpha}}, \tilde{\alpha} \in \widetilde{R}^0\}$  for *unitary* (respectively, *arbitrary*)  $\mathcal{R}$ . By *unitary*, we mean that  $\{\mathcal{R}_{\tilde{\alpha}}\}$  are defined for all  $\tilde{\alpha} \in \widetilde{R}$  and satisfy the conditions  $\mathcal{R}_{\tilde{\alpha}}\mathcal{R}_{-\tilde{\alpha}} = 1$ .

The standard Hecke-type examples of  $\mathcal{R}$ -matrices and the compatibility are as follows.

Let us assume that  $T_i$  and  $\widetilde{W}$  act in a vector space  $V$  and  $T_{\tilde{\alpha}} \stackrel{\text{def}}{=} \widetilde{w}T_i\widetilde{w}^{-1}$  does not depend on the choice of  $\widetilde{w}$  such that  $\widetilde{w}(\alpha_i) = \tilde{\alpha}$  for an arbitrary  $\tilde{\alpha} \in \widetilde{R}_+$ . Then  $\{\mathcal{R}_{\tilde{\alpha}} = s_{\tilde{\alpha}}T_{\tilde{\alpha}}\}$  is a *constant*  $\mathcal{R}$ -matrix.

An example of such  $V$  is the polynomial representation (see below) restricted to the affine Hecke algebra generated by  $\{T_i, i \geq 0\}$  with the usual action of  $\widetilde{W}$  on Laurent polynomials. In the abstract theory of quantum Knizhnik–Zamolodchikov equations, the action of  $\widetilde{w}$  through the projection  $\widetilde{W} \rightarrow W$  is used, i.e., with  $T_{[\alpha, \nu_{\alpha}j]}$  depending only on  $\alpha \in R$  (see [C2]).

Another (well-known) example is for the root system of type  $A_n$  only. We set  $T_{\alpha} = T_{ij}$  for  $\alpha = \epsilon_i - \epsilon_j$  and the so-called Baxter–Jimbo matrices  $T_{ij}$  in  $V_N \otimes V_N$  in the  $i, j$ -components of the tensor power  $V_N^{\otimes(n+1)}$  of the fundamental  $N$ -dimensional representation  $V_N$  of  $GL_N$ .

**Using intertwiners.** Let  $\Lambda_a$  ( $a \in Q$ ) be multiplicative *scalar* variables: they satisfy all the formulas for  $X_a, X_{\tilde{\alpha}}$  and are supposed to commute with  $\{T_{\tilde{\alpha}}\}$  defined via  $T_{\tilde{\alpha}} \stackrel{\text{def}}{=} \widetilde{w}T_i\widetilde{w}^{-1}$  as above. Here  $Q$  is the root lattice, but it can be replaced by  $P$  (or  $B$ ) if the extended case is considered.

The main properties of the intertwiners  $\Phi$  and  $G$  can be reformulated (generalized) as follows. The collections  $\mathcal{R}$  and  $\mathcal{G}$  given by

$$(5.22) \quad \begin{aligned} \mathcal{R}_{\tilde{\alpha}} &\stackrel{\text{def}}{=} s_{\tilde{\alpha}}T_{\tilde{\alpha}} + (t_{\alpha}^{1/2} - t_{\alpha}^{-1/2})/(\Lambda_{\tilde{\alpha}} - 1), \quad \tilde{\alpha} \in \widetilde{R}_+, \\ \mathcal{G}_{\tilde{\alpha}} &\stackrel{\text{def}}{=} \mathcal{R}_{\tilde{\alpha}}(t_{\alpha}^{1/2} + (t_{\alpha}^{1/2} - t_{\alpha}^{-1/2})/(\Lambda_{\tilde{\alpha}} - 1))^{-1}, \end{aligned}$$

are *non-constant*  $\text{End}(V)$ -valued  $\mathcal{R}$ -matrices. Their extensions to  $\widehat{W}$  are given by  $\pi_r \mapsto 1$ .

Note that one can use the formulas  $T_{\tilde{\alpha}} = \widetilde{w}T_i\widetilde{w}^{-1}$  to define  $T_{\tilde{\alpha}}$  for all  $\tilde{\alpha}$ , non-necessarily positive. Then  $\mathcal{G}$  is unitary,  $\mathcal{G}_{\tilde{\alpha}}\mathcal{G}_{-\tilde{\alpha}} = 1$ ,  $\mathcal{R}$  is not.

The (algebraically non-trivial) fact that  $\mathcal{R}, \mathcal{G}$  are  $\mathcal{R}$ -matrices can be deduced from the consideration of the intertwiners  $\Phi, G$  in the representation of  $\mathcal{H}^b$  induced from a *generic* character  $\Lambda : X_a \mapsto \Lambda_a$  of  $\mathbb{Q}_{q,t}[X]$ ; they are applied to the cyclic vector. The space of this induced representation is naturally isomorphic to the group algebra of the affine Hecke algebra generated by  $\{T_i, i \geq 0\}$ . Note that these  $\mathcal{R}$ -matrices are functions with the values in any  $\mathcal{H}^b$ -modules  $V$  with an action of  $\widetilde{W}$  such that the elements  $T_{\tilde{\alpha}} = \tilde{w}T_i\tilde{w}^{-1}$  are well defined (do not depend on the choice of  $\tilde{w}$  such that  $\tilde{w}(\alpha_i) = \tilde{\alpha}$ ).

Theorem 5.2 readily gives that the collection  $\{\mathcal{G}_{\tilde{\alpha}}^0 \stackrel{\text{def}}{=} s_{\tilde{\alpha}}T_{\tilde{\alpha}} = \mathcal{R}_{\tilde{\alpha}}^0\}$  is an  $\mathcal{R}$ -matrix that is *completely compatible* with  $\mathcal{G}$  and also *partially compatible* with  $\mathcal{R}$  from (5.22) for any root subsystem  $\tilde{R}^0$  provided that

$$\Lambda_{\tilde{\alpha}} = 1 \implies \tilde{\alpha} \in \tilde{R}^0.$$

We must also impose the inequality  $\Lambda_{\tilde{\alpha}} \neq t_{\alpha}^{\pm 1}$  for  $\mathcal{G}$  whenever  $\tilde{\alpha} \notin \tilde{R}^0$ .

We note that the case of  $\mathcal{G}$  here is related to Proposition 8.11 from [L] (in the affine case). The considerations from Section 6 from this paper are similar to our “abstract” approach to the relative Bruhat ordering and *completely compatible*  $\mathcal{R}$ -matrices.

**Comment.** (i) There is a variant of this construction when  $\mathcal{G}$  is always used instead of  $\mathcal{R}$  as  $\Lambda_{\tilde{\alpha}} \neq 1, t_{\alpha}^{\pm 1}$ , i.e., we make  $\tilde{\mathcal{R}}$  as unitary as possible. It gives an example of *partially compatible triple*  $\{\mathcal{G}^0, \mathcal{R}, \mathcal{G}\}$ .

(ii) The *trivial*  $\{\mathcal{G}_{\tilde{\alpha}}^0 = 1, \tilde{\alpha} \in \tilde{R}^0\}$  is *completely compatible* with  $\mathcal{G}$ ; the corresponding  $\tilde{\mathcal{G}}_{\tilde{w}}$  equals  $\mathcal{G}_{\tilde{w}^\circ}$  for the minimal element  $\tilde{w}^\circ$  from Theorem 4.2,(d). Thus the *complete compatibility*, equivalently, the fact that  $\tilde{\mathcal{G}}_{\tilde{w}}$  does not depend on the choice of the reduced decomposition of  $\tilde{w}$ , is not obvious even in this (simplest) case.

(iii) The  $\Lambda$ -factors that appear when the Coxeter transformations of type (5.17) are used in the products for  $\tilde{\mathcal{R}}$  are scalar and easy to control; so the difference between  $\mathcal{R}$  and  $\mathcal{G}$  is simply a matter of normalization. It becomes much more involved when the intertwiners  $\tilde{\Phi}_{\tilde{w}}$  are used to define  $\tilde{\mathcal{R}}$ . Replacing  $\Phi$  by the normalized intertwiners  $G$  is possible only in DAHA-modules where non-invertible intertwiners do not appear.

□

**A connection with virtual links.** If  $\tilde{R}^0$  is not involved and one can uses  $\mathcal{G}^0 = \{s_{\tilde{\alpha}}T_{\tilde{\alpha}}\}$  instead of  $\mathcal{G}$  for any roots, then the rank two relations for  $\{\mathcal{G}^0, \mathcal{G}\}$  are essentially the axioms of the *virtual links* generalized

to arbitrary reduced affine root systems. Here  $\mathcal{G}^0$  serves usual links,  $\mathcal{G}$  stays for virtual links. In the DAHA theory, the combinations like  $\mathcal{G}_{12}^0 \mathcal{G}_{13}^0 \mathcal{G}_{23}$  (with exactly two  $\mathcal{G}^0$ ) cannot appear. These ones also do not satisfy the triangle (Reidemeister III) property in the theory of virtual links.

Recall that  $\mathcal{G}$  is unitary and the transformations (5.17) are allowed. In the context of the  $\mathcal{R}$ -matrices, (5.17) is the *special relation* of type  $A_2$  from (5.21) with  $\mathcal{G}^0$  is in the middle and  $\mathcal{G}$  at the edges. If  $\mathcal{R}$  is used instead of  $\mathcal{G}$ , then (5.21) does not hold, so the general DAHA-theory, generally, requires more sophisticated approach.

In a more abstract way, the algebra formally generated by DAHA and the intertwining operators (defined for  $X$  or for  $Y$ , normalized or not) seems the key object. It is challenging to understand its “topological meaning”. We also note that the subcategory of semisimple DAHA-modules may have something to do with the invariants of virtual links and the “categorification”.

## 6. POLYNOMIAL REPRESENTATION

From now on we will switch from  $\mathcal{H}$  to an **intermediate subalgebra**  $\mathcal{H}^b \subset \mathcal{H}$  with  $P$  replaced by a lattice  $B$  between  $Q$  and  $P$  (see [C9]). Respectively,  $\Pi$  is changed to the preimage  $\Pi^b$  of  $B/Q$  in  $\Pi$ . Generally, there can be two different lattices  $B_x$  and  $B_y$  for  $X$  and  $Y$ . We consider only  $B_x = B = B_y$  in the paper; respectively,  $a, b \in B$  in  $X_a, Y_b$ .

We also set  $\widehat{W}^b = B \cdot W \subset \widehat{W}$ , and replace  $m$  by the least  $\tilde{m} \in \mathbb{N}$  such that  $\tilde{m}(B, B) \subset \mathbb{Z}$  in the definition of the  $\mathbb{Q}_{q,t}$ .

The relations (iii,iv) from Definition 5.1 (cf. [L]) read as:

$$(6.1) \quad T_i X_b - X_{s_i(b)} T_i = (t_i^{1/2} - t_i^{-1/2}) \frac{s_i(X_b) - X_b}{X_{\alpha_i} - 1}, \quad 0 \leq i \leq n.$$

Replacing  $X_b$  by  $Y_b^{-1}$  we obtain the dual “ $T - Y$ ” relations.

Note that  $\mathcal{H}^b$  and the polynomial representations (and their rational and trigonometric degenerations) are actually defined over  $\mathbb{Z}$  extended by the parameters of DAHA. However the field  $\mathbb{Q}_{q,t}$  will be sufficient in this paper.

**6.1. Definitions.** The Demazure-Lusztig operators are as follows:

$$(6.2) \quad T_i = t_i^{1/2} s_i + (t_i^{1/2} - t_i^{-1/2})(X_{\alpha_i} - 1)^{-1}(s_i - 1), \quad 0 \leq i \leq n;$$

they obviously preserve  $\mathbb{Q}[q, t^{\pm 1/2}][X_b]$ . We note that only the formula for  $T_0$  involves  $q$ :

$$(6.3) \quad \begin{aligned} T_0 &= t_0^{1/2} s_0 + (t_0^{1/2} - t_0^{-1/2})(X_0 - 1)^{-1}(s_0 - 1), \text{ where} \\ X_0 &= qX_\vartheta^{-1}, \quad s_0(X_b) = X_b X_\vartheta^{-(b, \vartheta)} q^{(b, \vartheta)}, \quad \alpha_0 = [-\vartheta, 1]. \end{aligned}$$

The map sending  $T_j$  to the corresponding operator from (6.2),  $X_b$  to  $X_b$  (see (5.2)) and  $\pi_r \mapsto \pi_r$  induces a  $\mathbb{Q}_{q,t}$ -linear homomorphism from  $\mathcal{H}^b$  to the algebra of linear endomorphisms of  $\mathbb{Q}_{q,t}[X]$ . This  $\mathcal{H}^b$ -module is faithful and remains faithful when  $q, t$  take any nonzero complex values assuming that  $q$  is not a root of unity. It will be called the **polynomial representation**; the notation is

$$\mathcal{V} \stackrel{\text{def}}{=} \mathbb{Q}_{q,t}[X_b] = \mathbb{Q}_{q,t}[X_b, b \in B].$$

The images of the  $Y_b$  are called the **difference Dunkl operators**. To be more exact, they must be called difference-trigonometric Dunkl operators, because there are also difference-rational Dunkl operators.

The polynomial representation is the  $\mathcal{H}^b$ -module induced from the one dimensional representation  $T_i \mapsto t_i^{1/2}$ ,  $Y_i \mapsto t_i^{1/2}$  of the affine Hecke subalgebra  $\mathcal{H}_Y^b = \langle T_i, Y_b \rangle$ . Here the following PBW-type theorem is used. For arbitrary nonzero  $q, t$ , any element  $H \in \mathcal{H}^b$  has a unique decomposition in the form

$$(6.4) \quad H = \sum_{w \in W} g_w f_w T_w, \quad g_w \in \mathbb{Q}_{q,t}[X_b], \quad f_w \in \mathbb{Q}_{q,t}[Y_b].$$

**Invariant form.** The definition is in terms of the **truncated theta function**

$$(6.5) \quad \mu = \mu^{(k)} = \prod_{\alpha \in R_+} \prod_{i=0}^{\infty} \frac{(1 - X_\alpha q_\alpha^i)(1 - X_\alpha^{-1} q_\alpha^{i+1})}{(1 - X_\alpha t_\alpha q_\alpha^i)(1 - X_\alpha^{-1} t_\alpha q_\alpha^{i+1})}.$$

It is considered as a Laurent series with the coefficients in  $\mathbb{Q}[t_\nu][[q_\nu]]$  for  $\nu \in \nu_R$ .

The constant term of a Laurent series  $f(X)$  will be denoted by  $\langle f \rangle$ . Then

$$(6.6) \quad \langle \mu \rangle = \prod_{\alpha \in R_+} \prod_{i=1}^{\infty} \frac{(1 - q^{(\rho_k, \alpha) + i\nu_\alpha})^2}{(1 - t_\alpha q^{(\rho_k, \alpha) + i\nu_\alpha})(1 - t_\alpha^{-1} q^{(\rho_k, \alpha) + i\nu_\alpha})}.$$

Recall that  $q^{(z,\alpha)} = q_\alpha^{(z,\alpha^\vee)}$ ,  $t_\alpha = q_\alpha^{k_\alpha}$ , so we can set here  $q^{(\rho_k,\alpha)+i\nu_\alpha} = q_\alpha^{(\rho_k,\alpha^\vee)+i}$ . This formula is equivalent to the Macdonald constant term conjecture from [M2], proved in complete generality in [C4].

Let  $\mu_\circ \stackrel{\text{def}}{=} \mu / \langle \mu \rangle$ . The coefficients of the Laurent series  $\mu_\circ$  are from the field of rationals  $\mathbb{Q}(q, t) \stackrel{\text{def}}{=} \mathbb{Q}(q_\nu, t_\nu)$ , where  $\nu \in \nu_R$ . Note that  $\mu_\circ^* = \mu_\circ$  for the involution

$$X_b^* = X_{-b}, \quad t^* = t^{-1}, \quad q^* = q^{-1}.$$

This involution is the restriction of the anti-involution  $\star$  from (5.10) to  $X$ -polynomials (and Laurent series). These two properties of  $\mu_\circ$  can be directly seen from the difference relations for  $\mu$ .

**Proposition 6.1.** *Setting,*

$$(6.7) \quad \langle f, g \rangle_\circ \stackrel{\text{def}}{=} \langle \mu_\circ f, g^* \rangle = \langle g, f \rangle_\circ^* \text{ for } f, g \in \mathbb{Q}(q, t)[X],$$

*the polynomial representation is  $\star$ -unitary:*

$$(6.8) \quad \langle H(f), g \rangle_\circ = \langle f, H^*(g) \rangle_\circ \text{ for } H \in \mathcal{H}, \quad f \in \mathbb{Q}_{q,t}[X].$$

**6.2. Macdonald polynomials.** There are two equivalent definitions of the **nonsymmetric Macdonald polynomials**, denoted by  $E_b(X) = E_b^{(k)}$  for  $b \in B$ . They belong to  $\mathbb{Q}(q, t)[X_a, a \in B]$  and, using the pairing  $\langle \cdot, \cdot \rangle$ , can be introduced by means of the conditions

$$(6.9) \quad E_b - X_b \in \oplus_{c \succ b} \mathbb{Q}(q, t)X_c, \quad \langle E_b, X_c \rangle_\circ = 0 \text{ for } B \ni c \succ b.$$

They are well defined because the pairing is nondegenerate (for generic  $q, t$ ) and form a basis in  $\mathbb{Q}(q, t)[X_b]$ .

This definition is due to Macdonald (for  $k_{\text{sht}} = k_{\text{lng}} \in \mathbb{Z}_+$ ), who extended Opdam's nonsymmetric polynomials introduced in the differential case in [O3] (Opdam mentions Heckman's unpublished lectures in [O3]). The general (reduced) case was considered in [C6].

Another approach is based on the  $Y$ -operators. We continue using the same notation  $X, Y, T$  for these operators acting in the polynomial representation.

**Proposition 6.2.** *The polynomials  $\{E_b, b \in B\}$  are unique (up to proportionality) eigenfunctions of the operators  $\{L_f \stackrel{\text{def}}{=} f(Y_1, \dots, Y_n), f \in$*

$\mathbb{Q}[X]\}$  acting in  $\mathbb{Q}_{q,t}[X]$  :

$$(6.10) \quad L_f(E_b) = f(q^{-b_\sharp})E_b \text{ for } b_\sharp \stackrel{\text{def}}{=} b - u_b^{-1}(\rho_k),$$

$$(6.11) \quad X_a(q^b) = q^{(a,b)}, \text{ where } a, b \in B, \quad u_b = \pi_b^{-1}b,$$

$u_b$  is from Proposition 1.3,  $b_\sharp = \pi_b((-\rho_k))$ .

□

The coefficients of the Macdonald polynomials are rational functions in terms of  $q_\nu, t_\nu$  (here either approach can be used).

**Symmetric polynomials.** Following Proposition 6.2, the **symmetric Macdonald polynomials**  $P_b = P_b^{(k)}$  can be introduced as eigenfunctions of the  $W$ -invariant operators  $L_f = f(Y_1, \dots, Y_n)$  defined for symmetric, i.e.,  $W$ -invariant, polynomials  $f$  as follows:

$$(6.12) \quad \begin{aligned} L_f(P_{b_-}) &= f(q^{-b_- + \rho_k})P_{b_-}, \quad b_- \in B_-, \\ P_{b_-} &= \sum_{b \in W(b_-)} X_b \pmod{\oplus_{c \succ b_-} \mathbb{Q}(q, t)X_c}, \end{aligned}$$

Here it suffices to take the **monomial symmetric functions**, namely,  $f_b = \sum_{c \in W(b)} X_c$  for  $b \in B_-$ . For minuscule  $b = -\omega_r$  and  $b = -\vartheta$ , the restrictions of the operators  $L_{f_b}$  to symmetric polynomials are the **Macdonald operators** from [M4, M3].

The  $P$ -polynomials are pairwise orthogonal with respect to  $\langle \cdot, \cdot \rangle_\circ$  as well as  $\{E\}$ . Since they are  $W$ -invariant,  $\mu$  can be replaced by the symmetric measure-function due to Macdonald:

$$(6.13) \quad \delta = \delta^{(k)} = \prod_{\alpha \in R_+} \prod_{i=0}^{\infty} \frac{(1 - X_\alpha q_\alpha^i)(1 - X_\alpha^{-1} q_\alpha^i)}{(1 - X_\alpha t_\alpha q_\alpha^i)(1 - X_\alpha^{-1} t_\alpha q_\alpha^i)}.$$

The corresponding pairing remains  $*$ -hermitian because the function  $\delta_\circ$  is  $*$ -invariant.

These polynomials were introduced in [M4, M3]. They were used for the first time in Kadell's unpublished work (classical root systems). In the one-dimensional case, they are due to Rogers.

The connection between  $E$  and  $P$  is as follows

$$(6.14) \quad \begin{aligned} P_{b_-} &= \mathbf{P}_{b_+} E_{b_+}, \quad b_- \in B_-, \quad b_+ = w_0(b_-), \\ \mathbf{P}_{b_+} &\stackrel{\text{def}}{=} \sum_{c \in W(b_+)} \prod_{\nu} t_\nu^{l_\nu(w_c)/2} T_{w_c}, \quad \text{where} \end{aligned}$$



$w_c \in W$  is the element of the least length such that  $c = w_c(b_+)$ ; the notation  $\mathbf{P}$  will be used if the summation is over all  $w$ . See [O3, M5, C6].

**6.3. Using intertwiners.** The  $Y$ -intertwiners serve as creation operators in the theory of nonsymmetric Macdonald polynomials. They are defined as follows:

$$(6.15) \quad \Psi_i = \tau_+(T_i) + \frac{t_i^{1/2} - t_i^{-1/2}}{Y_{\alpha_i}^{-1} - 1}, \quad i \geq 0, \quad P_r = \tau_+(\pi_r), \quad r \in O',$$

$$\Psi_{\hat{w}} = P_r \Psi_{i_l} \dots \Psi_{i_1} \text{ for reduced decompositions } \hat{w} = \pi_r s_{i_l} \dots s_{i_1}.$$

Recall that  $\tau_+(T_0) = X_0^{-1} T_0^{-1} = q^{-1} X_{\vartheta} T_0^{-1}$ ,  $\tau_+(\pi_r) = q^{-(\omega_r, \omega_r)/2} X_r \pi_r$ . The elements  $\Psi_{\hat{w}}$  are the images  $\tau_- \sigma(\Phi_{\hat{w}})$  of the  $X$ -intertwiners  $\Phi_{\hat{w}}$ .

Indeed, thanks to the relations  $\sigma(X_b) = Y_b^{-1}$ ,  $\tau_-(Y_b) = Y_b$ :

$$\tau_- \sigma(\Phi_i X_b) = \tau_- \sigma(X_c \Phi_i) \Rightarrow \tau_- \sigma(\Phi_i)(Y_b^{-1}) = Y_c^{-1} \tau_- \sigma(\Phi_i)$$

for  $c = s_i(b)$ . However,

$$\tau_- \sigma(\Phi_i) = \tau_+(T_i) + (t_i^{1/2} - t_i^{-1/2})(Y_{\alpha_i}^{-1} - 1)^{-1}$$

because  $\tau_- \sigma = \tau_+ \tau_-^{-1}$  and  $\tau_-$  preserves  $T_i$ . Similarly,  $\tau_- \sigma(\pi_r) = \tau_+(\pi_r)$ . We obtain that

$$\Psi_{\hat{w}} Y_b = Y_{\hat{w}(b)} \Psi_{\hat{w}}, \text{ where } Y_{[b,j]} \stackrel{\text{def}}{=} Y_b q^{-j}.$$

The same property holds for the normalized intertwiners

$$F_{\hat{w}} \stackrel{\text{def}}{=} \tau_- \sigma(G_{\hat{w}}),$$

which are *unitary*, i.e., induce a homomorphism from  $\widehat{W}$  to a proper localization of  $\mathcal{H}^b$ .

The automorphism  $\tau_-$  acts in  $\mathcal{V}$  and commutes with the  $Y$ -operators. The following proposition describes its action on the  $\Psi$ -intertwiners.

**Proposition 6.3.** (i) For generic  $q, t$  (or for arbitrary  $q, t$  provided that the polynomial  $E_b$  for  $b \in B$  is well defined),

$$(6.16) \quad \tau_-(E_b) = q^{-\frac{(b_-, b_-)}{2} + (b_-, \rho_k)} E_b \text{ for } P_- \ni b_- \in W(b).$$

(ii) For any  $q, t$  and  $Y_0 = q^{-1} Y_{\vartheta}^{-1}$ ,

$$(6.17) \quad \begin{aligned} \tau_-(T_i) &= T_i \quad (i > 0), \quad \tau_-(\tau_+(T_0)) = \tau_+(T_0)^{-1} Y_0, \\ \tau_-(\Psi_i) &= \Psi_i \quad (i > 0), \quad \tau_-(\Psi_0) = \Psi_0 Y_0 = Y_0^{-1} \Psi_0, \\ \tau_-(\tau_+(\pi_r)) &= q^{(\omega_r, \omega_r)/2} Y_r \tau_+(\pi_r) = q^{-(\omega_r, \omega_r)/2} \tau_+(\pi_r) Y_{r^*}^{-1}. \end{aligned}$$

*Proof.* These claims can be checked using directly the definitions. One can also identify  $\tau_-$  in  $\mathcal{V}$  with the operator of multiplication by  $\widetilde{\tau_-} = \gamma_y(1)^{-1} \gamma_y$  for the  $Y$ -Gaussian defined as follows:

$$(6.18) \quad \gamma_y \stackrel{\text{def}}{=} \sum_{b \in B} q^{(b,b)/2} Y_b, \quad \gamma_y(1) = \sum_{b \in B} q^{(b,b)/2} q^{(b, \rho_k)}.$$

Here we assume that  $0 < q < 1$  and use that  $\mathcal{V}$  is a union of finite dimensional spaces preserved by the  $Y$ -operators.

The irreducibility of  $\mathcal{V}$  for generic  $q, t$  is sufficient to conclude that  $\widetilde{\tau_-} = \tau_-$  because the conjugation by  $\widetilde{\tau_-}$  induces  $\tau_-$  in  $\mathcal{H}^p$ . This implies their coincidence for generic  $q$  (apart from roots of unity) and  $t_\nu$ , which results in (6.17) for *arbitrary*  $q, t$ .  $\square$

Setting

$$(6.19) \quad \Psi_i^b = \Psi_i(q^{b_\sharp}) = \tau_+(T_i) + (t_i^{1/2} - t_i^{-1/2})(X_{\alpha_i}(q^{b_\sharp}) - 1)^{-1},$$

$$(6.20) \quad F_i^b = (\Psi_i \psi_i^{-1})(q^{b_\sharp}) = \frac{\tau_+(T_i) + (t_i^{1/2} - t_i^{-1/2})(X_{\alpha_i}(q^{b_\sharp}) - 1)^{-1}}{t_i^{1/2} + (t_i^{1/2} - t_i^{-1/2})(X_{\alpha_i}(q^{b_\sharp}) - 1)^{-1}},$$

we come to the Main Theorem 5.1 from [C7].

**Theorem 6.4.** *Given  $c \in B$ ,  $0 \leq i \leq n$  such that  $(\alpha_i, c + d) > 0$ ,*

$$(6.21) \quad E_b q^{-(b,b)/2} = t_i^{1/2} \Psi_i^c(E_c) q^{-(c,c)/2} \quad \text{for } b = s_i((c)).$$

*If  $(\alpha_i, c + d) = 0$ , then*

$$(6.22) \quad \tau_+(T_i)(E_c) = t_i^{1/2} E_c, \quad 0 \leq i \leq n,$$

*which results in the relations  $s_i(E_c) = E_c$  as  $i > 0$ . For  $b = \pi_r((c))$ , where the indices  $r$  are from  $O'$ ,*

$$(6.23) \quad q^{-(b,b)/2 + (c,c)/2} E_b = \tau_+(\pi_r)(E_c) = X_{\omega_r} q^{-(\omega_r, \omega_r)/2} \pi_r(E_c).$$

*Also  $\tau_+(\pi_r)(E_c) \neq E_c$  for  $\pi_r \neq id$ , since  $\pi_r((c)) \neq c$  for any  $c \in B$  due to Lemma 1.5.*  $\square$

Using the standard Bruhat ordering and the set  $\mathcal{B}(\widehat{w})$  from Proposition 1.6, the theorem results in the following.

**Corollary 6.5.** *Given a reduced decomposition  $\pi_b = \pi_r s_{i_1} \cdots s_{i_l}$  and  $\widehat{w}' \in \mathcal{B}(\pi_b)$ , we define  $T_{\widehat{w}'}$  for the corresponding, possibly non-reduced, decomposition of  $\widehat{w}'$ . Then*

$$(6.24) \quad \tau_+(T_{\widehat{w}'})(1) \in \oplus_{\pi_{b'} \in \mathcal{B}(\pi_b)} \mathbb{Q}_{q,t} E_{b'} \subset \oplus_{a \geq b} \mathbb{Q}_{q,t} X_a.$$

*Proof.* We use (6.15), that is

$$\Psi_i = \tau_+(T_i) + \frac{t_i^{1/2} - t_i^{-1/2}}{Y_{\alpha_i}^{-1} - 1}, \quad P_r = \tau_+(\pi_r),$$

and that  $\{E_c\}$  are eigenfunctions of the  $Y$ -operators.  $\square$

We can now renormalize the  $E$ -polynomials as follows:

$$(6.25) \quad \widehat{E}_b \stackrel{\text{def}}{=} \tau_+(G_r G_{i_l}^{c_l} \dots G_{i_1}^{c_1})(1), \quad \text{where} \\ c_1 = 0, c_2 = s_{i_1}((c_1)), \dots, c_l = s_{i_l}((c_{l-1})) \quad \text{for } \pi_b = \pi_r s_{i_l} \dots s_{i_1}.$$

These polynomials do not depend on the particular choice of the decomposition of  $\pi_b$  (not necessarily reduced) and are proportional to  $E_b$  for all  $b \in B$ :

$$(6.26) \quad E_b q^{-(b,b)/2} = \prod_{1 \leq p \leq l} (t_{i_p}^{1/2} \phi_{i_p}(q^{c_p})) \widehat{E}_b \\ = \prod_{[\alpha, j] \in \lambda'(\pi_b)} \left( \frac{1 - q_\alpha^j t_\alpha X_\alpha(q^{\rho_k})}{1 - q_\alpha^j X_\alpha(q^{\rho_k})} \right) \widehat{E}_b.$$

The polynomials  $\widehat{E}_b$  are directly connected with the spherical polynomials.

**6.4. Spherical polynomials.** The following renormalization of the  $E$ -polynomials is of major importance in the Fourier analysis (see [C6]):

$$(6.27) \quad \mathcal{E}_b \stackrel{\text{def}}{=} E_b(X)(E_b(q^{-\rho_k}))^{-1}, \quad \text{where} \\ E_b(q^{-\rho_k}) = q^{(\rho_k, b_-)} \prod_{[\alpha, j] \in \lambda'(\pi_b)} \left( \frac{1 - q_\alpha^j t_\alpha X_\alpha(q^{\rho_k})}{1 - q_\alpha^j X_\alpha(q^{\rho_k})} \right),$$

$$(6.28) \quad \lambda'(\pi_b) = \{[\alpha, j] \mid [-\alpha, \nu_\alpha j] \in \lambda(\pi_b)\}.$$

We call them **spherical polynomials**. Explicitly (see (1.25)),

$$(6.29) \quad \lambda'(\pi_b) = \{[\alpha, j] \mid \alpha \in R_+, \\ -(b_-, \alpha^\vee) > j > 0 \text{ if } u_b^{-1}(\alpha) \in R_-, \\ -(b_-, \alpha^\vee) \geq j > 0 \text{ if } u_b^{-1}(\alpha) \in R_+\}.$$

Formula (6.27) is the Macdonald **evaluation conjecture** in the non-symmetric variant from [C6]. See [C5] for the symmetric evaluation conjecture.

Note that one has to consider only long  $\alpha$  (resp., short) if  $k_{\text{sh}} = 0$  (resp.,  $k_{\text{lg}} = 0$ ) in the  $\lambda'$ -set. All formulas involving  $\lambda$  or  $\lambda'$  have to be modified correspondingly in such case.

We have the following **duality formula** for  $b, c \in B$  :

$$(6.30) \quad \mathcal{E}_b(q^{c^\sharp}) = \mathcal{E}_c(q^{b^\sharp}), \quad b_\sharp = b - u_b^{-1}(\rho_k),$$

that is the main justification of the definition of  $\mathcal{E}_b$ .

Combining (6.26) and (6.27), we conclude that

$$(6.31) \quad \mathcal{E}_b = q^{(-\rho_k + b_-, b_-)/2} \widehat{E}_b.$$

See also [C7].

The proof of the duality formula (6.30) is based on the anti-involution  $\phi$  from (5.11):

$$\phi : X_b \mapsto Y_b^{-1} \mapsto X_b, \quad T_i \mapsto T_i \quad (1 \leq i \leq n), \quad q \mapsto q, \quad t \mapsto t.$$

Following [C5, C6] (see also (7.10) below), we define the **evaluation pairing** for  $f, g \in \mathcal{V}$ ,

$$(6.32) \quad \{f, g\} = \{L_{\iota(f)}(g(X))\} = \{L_{\iota(f)}(g(X))\}(q^{-\rho_k}), \\ \iota(X_b) = X_{-b} = X_b^{-1}, \quad \iota(z) = z \quad \text{for } z \in \mathbb{Q}_{q,t},$$

where  $L_f$  is from Proposition 6.2. This pairing is symmetric and induces  $\phi$  in  $\mathcal{H}^b$ .

As an application of (6.31), we obtain that, given  $b \in B$ , the spherical polynomial  $\mathcal{E}_b$  is well defined for  $q, t \in \mathbb{C}^*$  if

$$(6.33) \quad \prod_{[\alpha, j] \in \lambda'(\pi_b)} (1 - q_\alpha^j t_\alpha X_\alpha(q^{\rho_k})) \neq 0.$$

Similarly (see [C7], Corollary 5.3), the polynomial  $E_b$  exists if

$$\prod_{[\alpha, j] \in \lambda'(\pi_b)} (1 - q_\alpha^j X_\alpha(q^{\rho_k})) \neq 0.$$

If  $b \in B_-$  and the latter inequality holds for  $b_+ = w_0(b) \in B_+$ , then the symmetric polynomials  $P_b$  is well defined.

**Proof of the evaluation formula.** Another application of the intertwiners is a different approach to the evaluation formula (6.27); it is especially important when  $q, t$  are arbitrary (non-generic) and  $\mathcal{V}$  is not supposed semisimple. Note that the other two known proofs of the evaluation formula are based respectively on the duality and the

technique of shift operators (see [C5, C6]). The following proposition readily gives (6.27).

**Proposition 6.6.** (i) *Let us assume that  $E'_c$  is a  $Y$ -eigenvector satisfying (6.10) for  $c_\#$ . We will introduce  $E'_b$  using either formula (6.21) for  $b = s_i((c))$ , where  $i \geq 0$  and  $l(\pi_b) = 1 + l(\pi_c)$ , or using (6.23) for  $b = \pi_r((c))$ . Then  $E'_b$  satisfies (6.10) for  $b_\#$  and*

$$(6.34) \quad q^{-(\rho_k, b_-)} E'_b(q^{-\rho_k}) = q^{-(\rho_k, c_-)} E'_c(q^{-\rho_k}) \frac{1 - q_{\alpha}^{(\tilde{\alpha}^\vee, c_- + d)} t_{\alpha} X_{\alpha}(q^{\rho_k})}{1 - q_{\alpha}^{(\tilde{\alpha}^\vee, c_- + d)} X_{\alpha}(q^{\rho_k})},$$

$$\tilde{\alpha} = u_c(\alpha_i), \quad \alpha = u_c(\alpha_i) \text{ for } i > 0, \quad \alpha = u_c(-\vartheta) \text{ for } i = 0,$$

$$(6.35) \quad q^{-(\rho_k, b_-)} E'_b(q^{-\rho_k}) = q^{-(\rho_k, c_-)} E'_c(q^{-\rho_k}) \text{ for } b = \pi_r((c)).$$

(ii) *More generally, let  $\tilde{E} = \Psi_i(E)$  for  $E \in \mathcal{V}$ ,  $i \geq 0$ , assuming that  $(Y_{\alpha_i} - 1)^{-1}(E)$  is well defined. Then*

$$(6.36) \quad \tilde{E}(q^{-\rho_k}) = \left( \left( \frac{t_i^{1/2} Y_{\alpha_i}^{-1} - t_i^{-1/2}}{Y_{\alpha_i}^{-1} - 1} \right) (E) \right) (q^{-\rho_k}) \text{ for } i > 0,$$

$$\tilde{E}(q^{-\rho_k}) = \left( \left( \frac{t_0^{1/2} Y_{\alpha_0}^{-1} - t_0^{-1/2}}{Y_{\alpha_0}^{-1} - 1} Y_{\vartheta}^{-1} \right) (E) \right) (q^{-\rho_k}) \text{ for } i = 0.$$

If  $\tilde{E} = P_r(E)$ , where  $P_r = \tau_+(\pi_r)$ , then  $\tilde{E}(q^{-\rho_k}) = (Y_r^{-1}(E))(q^{-\rho_k})$ .

*Proof.* We will move  $\Psi_i^c$  in  $\{1, E'_b\} = E'_b(q^{-\rho_k})$  to the first component using (5.12) and then back. In more detail, we need the following relations:

$$(6.37) \quad \phi(\tau_+(T_i)) = (\star \cdot \tau_+ \cdot \eta)(T_i) = \tau_+(T_i), \quad \phi(P_r) = X_r u_r^{-1}.$$

Recall that  $\tau_+(T_i) = T_i$  for  $i > 0$  and  $\tau_+(T_0) = X_0^{-1} T_0^{-1}$  for  $X_0 = q X_{\vartheta}^{-1}$ .

The  $X$ -rational functions  $\phi(\Psi_i^c)(1)$  and  $\phi(P_r)(1)$  can be readily calculated. Then we move them back to the second component, replacing  $X$  by  $Y^{-1}$  and applying the result to  $E'_c$ , which is a  $Y$ -eigenvector.

Claim (ii) is a natural generalization of (i). First, using (6.37),

$$(6.38) \quad \phi(\Psi_i) = \phi(\tau_+(T_i)) + \frac{t_i^{1/2} - t_i^{-1/2}}{Y_{\alpha_i}^{-1} - 1} = \tau_+(T_i) + \frac{t_i^{1/2} - t_i^{-1/2}}{X_{\alpha_i} - 1}.$$

Second, the images of the  $\Psi$ -intertwiners in the polynomial representation are

$$(6.39) \quad \phi(\Psi_i)(f) = \frac{t_i^{1/2} X_{\alpha_i} - t_i^{-1/2}}{X_{\alpha_i} - 1} s_i^x(f), \quad \phi(P_r)(f) = \pi_r^x(f), \quad \text{where} \\ s_i^x(f) = s_i(f) \text{ for } i > 0, \quad s_0^x(f) = X_\vartheta s_\vartheta(f), \quad \pi_r^x(f) = X_r u_r^{-1}(f).$$

Let us give an explicit demonstration for  $i = 0$ :

$$\begin{aligned} \phi(\Psi_0) &= X_0^{-1} \left( \frac{t_0^{1/2} X_0 - t_0^{-1/2}}{X_0 - 1} s_0 \right) \\ &= X_0^{-1} \frac{(t_0^{1/2} - t_0^{-1/2}) X_0}{X_0 - 1} + \frac{t_0^{1/2} - t_0^{-1/2}}{X_0 - 1} \\ &= \frac{t_0^{1/2} X_0 - t_0^{-1/2}}{X_0 - 1} X_0^{-1} s_0 = \frac{t_0^{1/2} X_0 - t_0^{-1/2}}{X_0 - 1} s_0 X_0. \end{aligned}$$

Third, let  $\tilde{E} = \Psi_i(E)$ . We set  $\delta_{i0} = 1, 0$  respectively for  $i = 0$  or  $i > 0$ . Then the evaluation  $\tilde{E}(q^{-\rho_k})$  equals

$$\begin{aligned} &\left\{ \left( \frac{t_i^{1/2} X_{\alpha_i} - t_i^{-1/2}}{X_{\alpha_i} - 1} s_i^x \right) (1), E \right\} = \left\{ \left( \frac{t_i^{1/2} X_{\alpha_i} - t_i^{-1/2}}{X_{\alpha_i} - 1} X_{\vartheta^{i0}}^{\delta_{i0}} \right) (1), E \right\} \\ &= \left\{ 1, \left( \frac{t_i^{1/2} Y_{\alpha_i}^{-1} - t_i^{-1/2}}{Y_{\alpha_i}^{-1} - 1} Y_{\vartheta}^{-\delta_{i0}} \right) (E) \right\} = \left( \frac{t_i^{1/2} Y_{\alpha_i}^{-1} - t_i^{-1/2}}{Y_{\alpha_i}^{-1} - 1} Y_{\vartheta}^{-\delta_{i0}} \right) (E) (q^{-\rho_k}). \end{aligned}$$

The evaluation of  $\tilde{E} = P_r(E)$  is calculated similarly.  $\square$

A variant of the same construction gives that

$$(6.40) \quad (T_w(E))(q^{-\rho_k}) = \prod_{\nu} t_{\nu}^{l_{\nu}(w)/2} E(q^{-\rho_k}) \quad \text{for } E \in \mathcal{V}, \quad w \in W,$$

where  $l(w)$  is the length of *nonaffine*  $w$ .

This formula can be readily used to deduce the formula for  $P_b(q^{-\rho_k})$  for the symmetric Macdonald polynomials  $P_b$  ( $b \in B_-$ ) from that for the nonsymmetric polynomials. See (6.14).

## 7. SHIFT-OPERATOR

In this section, we take  $B = P$ . By  $[H]_{\dagger}$ , we mean the restriction of the image of the element  $H$  acting in the polynomial representation  $\mathcal{V}$

to the subspace  $\mathcal{V}^W$  of  $W$ -invariant Laurent polynomials. The image of  $[H]_{\dagger}$  is in  $\mathcal{V}^W$  if  $H$  is  $W$ -invariant, i.e., belongs to

$$\mathcal{H}^W \stackrel{\text{def}}{=} \{A \in \mathcal{H} \mid T_i A = A T_i, \text{ for } i > 0\}.$$

We fix a subset  $v \subset \nu_R$  and introduce the **shift operator**:

$$(7.1) \quad \begin{aligned} \mathcal{S}_v^t &= (\mathcal{X}_v^t)^{-1} \mathcal{Y}_v^t, \quad \mathbf{S}_v^{q,t} = [\mathcal{S}_v^t]_{\dagger} = (\mathcal{X}_v^t)^{-1} [\mathcal{Y}_v^t]_{\dagger}, \\ \mathcal{X}_v^t &\stackrel{\text{def}}{=} \prod_{\alpha \in R_+}^{\nu_{\alpha} \in v} ((t_{\alpha} X_{\alpha})^{1/2} - (t_{\alpha} X_{\alpha})^{-1/2}), \\ \mathcal{Y}_v^t &= \prod_{\alpha \in R_+}^{\nu_{\alpha} \in v} (t_{\alpha} Y_{\alpha}^{-1})^{1/2} - (t_{\alpha} Y_{\alpha}^{-1})^{-1/2}. \end{aligned}$$

Note that the elements  $\mathcal{X}_v^t, \mathcal{Y}_v^t$  belong respectively to  $\mathbb{Z}_t[X_b], \mathbb{Z}_t[Y_b]$  for  $b \in P$ .

**7.1. Shift-formula.** The following proposition is from [C4], [C9]. The  $W$ -invariant operators  $L_f^{q,t}$  for  $W$ -invariant  $f$  and the *symmetric* Macdonald polynomials  $P_b^{q,t}(X)$  defined in (6.12) are used.

**Proposition 7.1.** *The operators  $\mathcal{S}_v^t$  and its restriction  $\mathbf{S}_v^{q,t}$  to  $\mathbb{Q}_{q,t}[X]^W$  are  $W$ -invariant and preserve the latter space. Provided that  $t_{\nu} = 1$  whenever  $\nu \notin v$ ,*

$$(7.2) \quad \mathbf{S}_v^{q,t} L_f^{q,t} = L_f^{q,tq_v} \mathbf{S}_v^{q,t} \text{ for } f \in \mathbb{C}[X]^W,$$

where  $tq_v = \{t_{\nu} q^{\nu}, t_{\nu'}\}$  for  $\nu \in v \not\equiv \nu'$ ,

$$(7.3) \quad \begin{aligned} \mathbf{S}_v^{q,t}(P_b^{q,t}) &= g_v^{q,t}(b) P_{b+\rho_v}^{q,tq_v}, \text{ for} \\ g_v^{q,t}(b) &\stackrel{\text{def}}{=} \prod_{\alpha \in R_+, \nu_{\alpha} \in v} (X_{\alpha}(q^{(\rho_k-b)/2}) - t_{\alpha} X_{\alpha}(q^{(b-\rho_k)/2})), \end{aligned}$$

$$(7.4)$$

where  $\rho_v = \sum_{\nu \in v} \rho_{\nu}$ ,  $P_c = 0$  for  $c \notin P_-$ .

We will also need the element

$$\overline{\mathcal{Y}}_v^t \stackrel{\text{def}}{=} \prod_{\alpha \in R_+, \nu_{\alpha} \in v} ((t_{\alpha} Y_{\alpha})^{1/2} - (t_{\alpha} Y_{\alpha})^{-1/2});$$

it is  $\mathcal{Y}$  with  $Y_{\alpha}^{\pm 1/2}$  replaced by  $Y_{\alpha}^{\mp 1/2}$  (or  $\mathcal{X}$  with  $Y$  instead of  $X$ ).

The product  $\overline{\mathcal{Y}}_v^t \mathcal{Y}_v^t$  is obviously a  $W$ -invariant polynomial. So we can apply the formulas in (6.12):

$$\begin{aligned}
 (7.5) \quad & (\overline{\mathcal{Y}}_v^t \mathcal{Y}_v^t)(P_b^{q,t}(X)) = \overline{g}_v^{q,t}(b) g_v^{q,t}(b) \text{ for} \\
 & \overline{g}_v^{q,t}(b) \stackrel{\text{def}}{=} \prod_{\alpha \in R_+, \nu_\alpha \in v} (t_\alpha^{-1} X_\alpha(q^{(b-\rho_k)/2}) - X_\alpha(q^{(\rho_k-b)/2})) \\
 & = \prod_{\alpha \in R_+, \nu_\alpha \in v} t_\alpha^{-1} q^{(\alpha, (b-\rho_k))/2} \prod_{\alpha \in R_+, \nu_\alpha \in v} (1 - t_\alpha q^{(\alpha, \rho_k-b)}) \\
 & = q^{(\rho_v, (b-\rho_k))} \prod_{\nu \in v} t_\nu^{-\mathfrak{a}_\nu} \prod_{\alpha \in R_+, \nu_\alpha \in v} (1 - t_\alpha q^{(\alpha, \rho_k-b)}),
 \end{aligned}$$

where  $\mathfrak{a}_\nu$  is the number of positive roots with  $\nu_\alpha = \nu$ .

Note the formula

$$\overline{g}_v^{q,t}(b) g_v^{q,t}(b) = \prod_{\alpha \in R, \nu_\alpha \in v} (t_\alpha^{1/2} X_\alpha(q^{(b-\rho_k)/2}) - t_\alpha^{-1/2} X_\alpha(q^{(\rho_k-b)/2})),$$

where the product is over *all* roots.

We are going to employ the duality and the Macdonald (symmetric) evaluation conjecture, the formula for the “principle value” of the Macdonald polynomials proved in [C9] (in the  $A$ -case, both were verified by Koornwinder). They read as:

$$\begin{aligned}
 (7.6) \quad & P_b(q^{c-\rho_k}) P_c(q^{-\rho_k}) = P_c(q^{b-\rho_k}) P_b(q^{-\rho_k}), \quad b, c \in P_-, \\
 & P_b(q^{-\rho_k}) = P_b(q^{+\rho_k}) = X_b(q^{\rho_k}) \prod_{\alpha \in R_+} \prod_{j=1}^{-(\alpha^\vee, b)} \left( \frac{1 - q_\alpha^{j-1} t_\alpha X_\alpha(q^{\rho_k})}{1 - q_\alpha^{j-1} X_\alpha(q^{\rho_k})} \right).
 \end{aligned}$$

As a matter of fact, the second formula follows from the first via the Pieri rules. See [C9]. The derivation of the evaluation formula from the Pieri rules in the case of  $A_n$  is due to Koornwinder.

**Norm-formulas via the evaluation ones.** It is important to note that the duality also results in the norm-formulas, including the celebrated Macdonald constant term conjecture.

The simplest known deduction of the norm-formulas is based on the intertwining operators acting on the *spherical polynomials* that are the *nonsymmetric* Macdonald polynomials taking the value 1 at  $q^{-\rho_k}$ . The intertwiners preserve the latter normalization up to certain simple factors. They act changing the norms of  $E$ -polynomials, but this is simple



to control. Thus, using intertwiners we naturally come to the norm-formulas in the *spherical normalization*, i.e., for spherical polynomials.

Now one can use the evaluation formulas (principle value formulas) to get the norm-formulas in the standard monic normalization of the leading terms of the  $E$ -polynomials. The last step is reproving the *symmetric* Macdonald norm conjecture (stated in [M2] and proven in [C4]); we apply the symmetrization and use the formulas for  $\{P_{b_-}\}$  in terms of  $\{E_b\}$ .

This procedure gives a transparent deduction of the norm formulas from the evaluation ones. However it does not clarify why the norms appear the products *very much similar* to those in the definition of the Macdonald measure.

The “conceptual” reason for this coincidence (which is a result of a straight calculation in the above deduction) is as follows.

The above calculation in terms of the intertwining operators is actually *equivalent* to the calculation of the *Fourier transform*, acting from  $\mathcal{V}$  to the  $\mathcal{H}^b$ -representation in delta-functions. It sends the spherical polynomials exactly to the corresponding *delta-functions* defined with respect to the discretization of the Macdonald measure (with the natural normalization). It was established in [C7] (see also [C6, C12]). Therefore the norms of the spherical polynomials practically coincide with the *values* of the Macdonald measure at the corresponding weights. This gives a complete clarification of the constant term conjecture and concludes (the corresponding part of) [M2].

**7.2. Formula for  $\overline{\mathcal{Y}}(\mathcal{X})$ .** Later on, the shift operator will be used only when the set  $v$  is the whole  $\nu_R$ . The suffix  $v$  will be dropped in the formulas, the indices  $\nu$  are arbitrary from  $\nu_R$ . For instance,  $tq_v = tq = \{t_\nu q_\nu, \nu \in \nu_R\}$ . We will also use  $\rho_\nu^\vee = (1/2) \sum_{\nu_\alpha = \nu} \alpha^\vee$ :

$$(7.7) \quad \begin{aligned} t^{\rho^\vee} &= \prod_{\nu \in \nu_R} t_\nu^{\rho_\nu^\vee}, \quad t^{(\alpha, \rho^\vee)} = \prod_{\nu \in \nu_R} t_\nu^{(\alpha, \rho_\nu^\vee)} = q^{(\alpha, \rho_k)}, \\ t^{\mathfrak{a}} &= \prod_{\nu \in \nu_R} t_\nu^{\mathfrak{a}_\nu} \quad \text{for } \mathfrak{a}_\nu = |\{\alpha \in R_+, \nu_\alpha = \nu\}|. \end{aligned}$$

**Main Theorem 7.2.** *Let  $v = \nu_R$  and  $b = -\rho$ . Using the notation  $q^{(\rho_k - b)} = (tq)^{\rho^\vee} = \prod_\nu (t_\nu q_\nu)^{\rho_\nu^\vee}$ , and*

$$(tq)^{(\cdot, \rho^\vee)} = \prod_\nu (t_\nu q_\nu)^{(\cdot, \rho_\nu^\vee)}, \quad \text{e.g., } (tq)^{(\rho, \rho^\vee)} = \prod_\nu (t_\nu q_\nu)^{(\rho, \rho_\nu^\vee)} = q^{(\rho, \rho + \rho_k)},$$

we come to the relation

$$\begin{aligned}
 (7.8) \quad \overline{\mathcal{Y}}^t(\mathcal{X}^t) &= C_{q,t} P_b^{q,t} \quad \text{for } C_{q,t} = \overline{g}^{q,t}(b) \\
 &= \prod_{\alpha \in R_+} (t_\alpha^{-1} (tq)^{-(\alpha, \rho^\vee)/2} - (tq)^{(\alpha, \rho^\vee)/2}) \\
 &= t^{-\mathfrak{A}} (tq)^{(\rho, \rho^\vee)} \prod_{\alpha \in R_+} (1 - t_\alpha (tq)^{(\alpha, \rho^\vee)}).
 \end{aligned}$$

Applying the evaluation formula for  $P_{-\rho}^{q,t}(X)$ ,

$$\begin{aligned}
 (7.9) \quad \overline{\mathcal{Y}}^t(\mathcal{X}^t)(q^{-\rho_k}) &= C_{q,t} P_{-\rho}^{q,t}(q^{-\rho_k}) = t^{-\mathfrak{A}} (t^2 q)^{-(\rho, \rho^\vee)} \Pi_{\tilde{R}}, \\
 \Pi_{\tilde{R}} &\stackrel{\text{def}}{=} \prod_{\alpha \in R_+} \left( (1 - t_\alpha (tq)^{(\alpha, \rho^\vee)}) \prod_{j=1}^{(\alpha^\vee, \rho)} \frac{(1 - q_\alpha^{j-1} t_\alpha t^{(\alpha, \rho^\vee)})}{(1 - q_\alpha^{j-1} t^{(\alpha, \rho^\vee)})} \right) \\
 &= \prod_{\alpha \in R_+} \left( (1 - q_\alpha^{k_\alpha + (\alpha^\vee, \rho + \rho_k)}) \prod_{j=1}^{(\alpha^\vee, \rho)} \frac{(1 - q_\alpha^{j-1 + k_\alpha + (\alpha^\vee, \rho_k)})}{(1 - q_\alpha^{j-1 + (\alpha^\vee, \rho_k)})} \right), \\
 &\text{where } t^{-\mathfrak{A}} (t^2 q)^{-(\rho, \rho^\vee)} = q^{-\sum_\nu \nu k_\nu \mathfrak{A}_\nu} q^{-(\rho, \rho + 2\rho_k)}.
 \end{aligned}$$

*Proof.* The choice  $b = -\rho$  makes  $\mathbf{S}^{q,t}(P_b^{q,t})$  a constant, which equals to  $g^{q,t}(b)$ . Therefore  $\mathcal{Y}^t(P_b^{q,t}) = g^{q,t}(b) \mathcal{X}^t$  and

$$\overline{\mathcal{Y}}^t \mathcal{Y}^t(P_b^{q,t}) = g^{q,t}(b) \overline{\mathcal{Y}}^t(\mathcal{X}^t).$$

Thus  $\overline{\mathcal{Y}}^t(\mathcal{X}^t) = (g^t)^{-1} \overline{\mathcal{Y}}^t \mathcal{Y}^t(P_b^{q,t}) =$

$$\begin{aligned}
 &= \overline{g}^{q,t}(b) g^{q,t}(b) (g^{q,t}(b))^{-1} P_b^{q,t} = \overline{g}^{q,t}(b) P_b^{q,t} \\
 &= \prod_{\alpha \in R_+} (t_\alpha^{-1} X_\alpha(q^{(b-\rho_k)/2}) - X_\alpha(q^{(\rho_k-b)/2})) P_b^{q,t} \\
 &= \prod_{\alpha \in R_+} (t_\alpha^{-1} q^{(\alpha, b-\rho_k)/2} - q^{(\alpha, \rho_k-b)/2}) P_b^{q,t} \\
 &= \prod_{\alpha \in R_+} t_\alpha^{-1} q^{(\alpha, (b-\rho_k))/2} \prod_{\alpha \in R_+} (1 - t_\alpha q^{(\alpha, \rho_k-b)}) P_b^{q,t} \\
 &= \prod_{\nu \in \nu_R} t_\nu^{-\mathfrak{A}_\nu} (t_\nu q_\nu)^{-(\rho, \rho^\vee)} \prod_{\alpha \in R_+} \left( (1 - t_\alpha \prod_{\nu \in \nu_R} (t_\nu q_\nu)^{(\alpha, \rho^\vee)}) \right) P_b^{q,t}.
 \end{aligned}$$

The “principle value” of  $P_b$  as  $b = -\rho$  equals

$$\begin{aligned}
P_{-\rho}(q^{-\rho_k}) &= q^{-(\rho, \rho_k)} \prod_{\alpha \in R_+} \prod_{j=1}^{(\alpha^\vee, \rho)} \left( \frac{1 - q_\alpha^{j-1} t_\alpha q_\alpha^{(\alpha^\vee, \rho_k)}}{1 - q_\alpha^{j-1} q_\alpha^{(\alpha^\vee, \rho_k)}} \right) \\
&= \left( \prod_{\nu \in \nu_R} t_\nu^{-(\rho, \rho_\nu^\vee)} \right) \prod_{\alpha \in R_+} \prod_{j=1}^{(\alpha^\vee, \rho)} \frac{1 - q_\alpha^{j-1} t_\alpha \prod_{\nu \in \nu_R} t_\nu^{(\alpha, \rho_\nu^\vee)}}{1 - q_\alpha^{j-1} \prod_{\nu \in \nu_R} t_\nu^{(\alpha, \rho_\nu^\vee)}} \\
&= t^{-(\rho, \rho^\vee)} \prod_{\alpha \in R_+} \prod_{j=1}^{(\alpha^\vee, \rho)} \frac{1 - q_\alpha^{j-1} t_\alpha t^{(\alpha, \rho^\vee)}}{1 - q_\alpha^{j-1} t^{(\alpha, \rho^\vee)}}.
\end{aligned}$$

Combining this formula with the previous one, we obtain:

$$\begin{aligned}
C_{q,t} \cdot P_{-\rho}(q^{-\rho_k}) &= \prod_{\nu \in \nu_R} t_\nu^{-\mathfrak{a}_\nu} (t_\nu q_\nu)^{-(\rho, \rho_\nu^\vee)} \prod_{\alpha \in R_+} (1 - t_\alpha \prod_{\nu \in \nu_R} (t_\nu q_\nu)^{(\alpha, \rho_\nu^\vee)}) \\
&\cdot \prod_{\nu \in \nu_R} t_\nu^{-(\rho, \rho_\nu^\vee)} \prod_{\alpha \in R_+} \prod_{j=1}^{(\alpha^\vee, \rho)} \left( \frac{1 - q_\alpha^{j-1} t_\alpha \prod_{\nu \in \nu_R} t_\nu^{(\alpha, \rho_\nu^\vee)}}{1 - q_\alpha^{j-1} \prod_{\nu \in \nu_R} t_\nu^{(\alpha, \rho_\nu^\vee)}} \right) \\
&= t^{-\mathfrak{a}} (t^2 q)^{-(\rho, \rho^\vee)} \prod_{\alpha \in R_+} (1 - t_\alpha (tq)^{(\alpha, \rho^\vee)}) \prod_{\alpha \in R_+} \prod_{j=1}^{(\alpha^\vee, \rho)} \left( \frac{(1 - q_\alpha^{j-1} t_\alpha t^{(\alpha, \rho^\vee)})}{(1 - q_\alpha^{j-1} t^{(\alpha, \rho^\vee)})} \right).
\end{aligned}$$

□

Generalizing the principle value for the  $X$ -polynomials, that is at  $q^{-\rho_k}$ , we set

$$(7.10) \quad \{H\} \stackrel{\text{def}}{=} H(1)(q^{-\rho_k}) \text{ for an operator } H \text{ acting in } \mathcal{V}.$$

Applying formulas (3.5)-(3.7) from Key Lemma 3.3 from [C5], we come to the following connection of the values of the symmetric Macdonald polynomials at  $q^{-\rho_k}$  as  $k \mapsto k+1$ . Here the evaluation formula is used too.

**Corollary 7.3.** *Let  $P = P_b^{q,t}$  for  $b \in P_-$  be the Macdonald polynomial for  $t$  and  $P_{b+\rho}^{q,tq}$  the one for  $tq$ ; it is zero if  $b + \rho \notin P_-$ . Then*

$$\begin{aligned}
(7.11) \quad \{\overline{\mathcal{Y}}^t \mathcal{X}^t\} P_{b+\rho}^{q,tq}(q^{-\rho_{k+1}}) &= \overline{g}^{q,t}(b) P_b^{q,t}(q^{-\rho_k}) \\
&= \prod_{\alpha \in R_+} (t_\alpha^{-1} X_\alpha(q^{(b-\rho_k)/2}) - X_\alpha(q^{(\rho_k-b)/2})) P_b^{q,t}(q^{-\rho_k}),
\end{aligned}$$

where  $\{\overline{\mathcal{Y}}^t \mathcal{X}^t\} = \overline{\mathcal{Y}}^t(\mathcal{X}^t)(q^{-\rho_k})$ . Equivalently,

$$\begin{aligned}
 (7.12) \quad & P_{-\rho}^{q,t}(q^{-\rho_k}) P_{b+\rho}^{q,tq}(q^{-\rho_{k+1}}) \\
 &= \prod_{\alpha \in R_+} \frac{t_{\alpha}^{-1} X_{\alpha}(q^{(-\rho_k+b)/2}) - X_{\alpha}(q^{(\rho_k-b)/2})}{t_{\alpha}^{-1} X_{\alpha}(q^{(-\rho_k-\rho)/2}) - X_{\alpha}(q^{(\rho_k+\rho)/2})} P_b^{q,t}(q^{-\rho_k}) \\
 &= q^{(b+\rho, \rho)} \prod_{\alpha \in R_+} \frac{1 - q_{\alpha}^{k_{\alpha} + (\alpha^{\vee}, \rho_k - b)}}{1 - q_{\alpha}^{k_{\alpha} + (\alpha^{\vee}, \rho_k + \rho)}} P_b^{q,t}(q^{-\rho_k}).
 \end{aligned}$$

□

**7.3. Rational limit.** Setting  $k_{\text{sht}} = k = \nu_{\text{lng}} k_{\text{lng}}$ ,  $t_{\text{sht}} = t = t_{\text{lng}}$ , the *Coxeter exponents* are defined from the formula

$$(7.13) \quad \prod_{\alpha \in R_+} \frac{1 - t^{1+(\alpha, \rho^{\vee})}}{1 - t^{(\alpha, \rho^{\vee})}} = \prod_{i=1}^n \frac{1 - t^{m_i+1}}{1 - t}, \quad 2\rho^{\vee} = \sum_{\alpha \in R_+} \alpha^{\vee}.$$

In the simply-laced case, the products  $\Pi_{\tilde{R}}$  and  $C_{q,t} P_{-\rho}^{q,t}(q^{-\rho_k})$  can be expressed in terms of the Coxeter exponents, Namely,  $\Pi_{\tilde{R}} =$

$$\begin{aligned}
 & \prod_{\alpha \in R_+} (1 - t(tq)^{(\alpha, \rho)}) \prod_{j=1}^{(\alpha, \rho)} \left( \frac{(1 - q^{j-1} t^{1+(\alpha, \rho)})}{(1 - q^{j-1} t^{(\alpha, \rho)})} \right) = \prod_{i=1}^n \frac{\prod_{j=0}^{m_i} (1 - q^j t^{m_i+1})}{1 - t}, \\
 (7.14) \quad & C_{q,t} P_{-\rho}^{q,t}(q^{-\rho_k}) = t^{-|R_+|} (t^2 q)^{-(\rho, \rho)} \prod_{i=1}^n \frac{\prod_{j=0}^{m_i} (1 - q^j t^{m_i+1})}{1 - t}.
 \end{aligned}$$

**Comment.** (i) Here, technically, the product  $\prod_{\alpha \in R_+} (1 - t(tq)^{(\alpha, \rho)})$  ensures the cancelation of the binomials in the denominator; it is not needed in (7.13). The process of such cancelation is not quite immediate even in the simply-laced case. It requires a combinatorial reformulation of (7.13) in terms of the sequence of the numbers of positive roots  $\alpha$  of given  $(\alpha, \rho^{\vee})$ , the *height*; cf. Lemma 12.3.

(ii) In our approach, the right-hand side automatically contains  $|R_+|$  extra factors in the numerator, which corresponds to the classical formula  $m_1 + \dots + m_n = |R_+|$ . The latter requires (simple) calculating the leading terms in (7.13). Another famous formula  $(m_i + 1) \cdots (m_n + 1) = |W|$ , which is immediate from the Poincaré polynomial interpretation

of (7.13), becomes the formula for the product of *rational exponents*, a counterpart of (7.14) for the rational DAHA.

(iii) It is not difficult to calculate  $\Pi_{\tilde{R}}$  in the non-simply-laced case under  $k_{\text{sht}} = k = \nu_{\text{lng}} k_{\text{lng}}$ . The structure of the formula is practically the same as in (7.14), but the indices  $j$  satisfy some non-trivial divisibility conditions and can become greater than  $m_i$ . We note that there is no significant simplification of  $\Pi_{\tilde{R}}$  under the substitution  $k_{\text{sht}} = k_{\text{lng}}$  (without  $\nu$ ).  $\square$

**Poincaré polynomial.** Let us recall the generalization of (7.13) to the case of two different  $k$ -parameters from [M1] and its relation to the Poincaré polynomial of  $W$ . We use the notion of the partial length  $l_\nu(w)$  from (1.8), where  $\nu \in \nu_R$ . One has:

$$(7.15) \quad \Pi_R \stackrel{\text{def}}{=} \sum_{w \in W} \prod_{\nu} t_\nu^{l_\nu(w)} = \prod_{\alpha \in R_+} \frac{1 - q_\alpha^{k_\alpha + (\alpha^\vee, \rho_k)}}{1 - q_\alpha^{(\alpha^\vee, \rho_k)}}.$$

See [C4] for the proof of this formula via the  $r$ -matrices and its applications to the Macdonald norm conjecture.

Let us list the counterparts of (7.13) in the non-simply-laced cases. Actually, they can be readily obtained from the formulas for the affine exponents considered below by picking the binomials without  $q^j$  for  $j > 0$ , i.e., the binomials given entirely in terms of  $t_{\text{sht}} = q^{k_{\text{sht}}}$  and  $t_{\text{lng}} = q_{\text{lng}}^{k_{\text{lng}}}$ . In the  $B - C$  cases,

$$(7.16) \quad B_n : \quad \Pi_R = \prod_{m=0}^{n-1} \left( \frac{1 - t_{\text{lng}}^{m+1}}{1 - t_{\text{lng}}} \right) \left( \frac{1 - t_{\text{lng}}^{2m} t_{\text{sht}}^2}{1 - t_{\text{lng}}^m t_{\text{sht}}} \right),$$

$$(7.17) \quad C_n : \quad \Pi_R = \prod_{m=0}^{n-1} \left( \frac{1 - t_{\text{sht}}^{m+1}}{1 - t_{\text{sht}}} \right) \left( \frac{1 - t_{\text{sht}}^{2j} t_{\text{lng}}^2}{1 - t_{\text{sht}}^m t_{\text{lng}}} \right).$$

These formulas are  $t_{\text{lng}} \leftrightarrow t_{\text{sht}}$ -dual to each other, which readily follows from the interpretation of  $\Pi_R$  in terms of the Poincaré polynomial. The same symmetry holds in the formulas below. In the case of  $G_2$ ,

$$(7.18) \quad G_2 : \quad \Pi_R = \frac{(1 - t_{\text{lng}}^2)(1 - t_{\text{sht}}^2)(1 - t_{\text{lng}}^3 t_{\text{sht}}^3)}{(1 - t_{\text{lng}})(1 - t_{\text{sht}})(1 - t_{\text{lng}} t_{\text{sht}})}.$$

In the case of  $F_4$ ,

$$(7.19) \quad F_4 : \Pi_R = \frac{(1 - t_{\text{lng}}^2)(1 - t_{\text{lng}}^3)(1 - t_{\text{sht}}^2)(1 - t_{\text{sht}}^3)}{(1 - t_{\text{lng}})(1 - t_{\text{lng}})(1 - t_{\text{sht}})(1 - t_{\text{sht}})} \\ \times \frac{(1 - t_{\text{lng}}^4 t_{\text{sht}}^2)(1 - t_{\text{lng}}^2 t_{\text{sht}}^4)(1 - t_{\text{lng}}^4 t_{\text{sht}}^4)(1 - t_{\text{lng}}^6 t_{\text{sht}}^6)}{(1 - t_{\text{lng}}^2 t_{\text{sht}})(1 - t_{\text{lng}} t_{\text{sht}}^2)(1 - t_{\text{lng}} t_{\text{sht}})(1 - t_{\text{lng}}^3 t_{\text{sht}}^3)}.$$

**Rational limit.** In the rational limit  $q = e^h, h \rightarrow 0$ , (7.9) results in the formula from Theorem 9.8 [O2] (the simply-laced case) and from Theorem 4.11 [DJO], where the root systems  $B_n, F_4$  and  $I_2(2m)$  were considered. In the rational setting, the limiting formulas for  $B$  and  $C$  are obtained from each other by the transposition  $k_{\text{lng}} \leftrightarrow k_{\text{sht}}$ ;  $G_2 = I_2(6)$ .

For the sake of concreteness, let us describe here the limiting procedure in the simply-laced case only. Let  $Y_b = e^{-\sqrt{h}\tilde{y}_b}$ ,  $X_b = e^{\sqrt{h}x_b}$ . Then the  $h$ -leading term of  $\tilde{y}_b$  becomes the **Dunkl operator** [D]:

$$y_b = \partial_b + \sum_{\alpha \in R_+} \frac{k(b, \alpha)}{x_\alpha} (1 - s_\alpha) \quad \text{for } \partial_b(x_c) = (b, c),$$

acting in the polynomial representation  $\mathbb{C}[x_b]$ . The limit of the shift operator  $\mathcal{S}$  is

$$\prod_{\alpha \in R_+} x_\alpha^{-1} \prod_{\alpha \in R_+} y_\alpha.$$

The  $h$ -leading term of the expression  $\overline{\mathcal{Y}}(\mathcal{X})$  is

$$(-h)^{|R_+|} \left( \prod_{\alpha \in R_+} y_\alpha \right) \left( \prod_{\alpha \in R_+} x_\alpha \right),$$

where we apply the  $y$ -operator to the  $x$ -polynomial. It is a *constant*, so the “principle value”, which is taken at zero in the rational theory, simply coincides with it.

The leading term of the product in the right-hand side of (7.14) can be readily calculated; it is

$$(-h)^{|R_+|} |W| \prod_{i=1}^n \prod_{j=1}^{m_i} (j + (m_i + 1)k).$$

We use here that the product  $\prod_{i=1}^n (1 - t^{m_i+1})/(1 - t)$  tends to  $|W|$ .

We arrive at the following formula due to Opdam.

**Corollary 7.4.** For  $y_b = \partial_b + \sum_{\alpha \in R_+} \frac{k(b, \alpha)}{x_\alpha} (1 - s_\alpha)$ ,

$$(7.20) \quad \left( \prod_{\alpha \in R_+} y_\alpha \right) \left( \prod_{\alpha \in R_+} x_\alpha \right) = |W| \prod_{i=1}^n \prod_{j=1}^{m_i} (j + (m_i + 1)k),$$

□

**Comment.** Formula (7.20) was proved in [O1] in the crystallographic case and then extended in [O2, DJO] to groups generated by complex reflections; its relation to the reducibility of the rational polynomial representation was established in the latter paper. See also paper [DO]. We obtain it as a rational limit of the  $q, t$ -formula; the rational limit is an entirely algebraic procedure.

Our approach via the *affine exponents* results in a direct link to the defining product formula for the *Coxeter exponents* in (7.13); see also (7.16, 7.17, 7.18, 7.19). This approach establishes a relation with the  $p$ -adic theory (Macdonald-Matsumoto) corresponding to the limit  $q \rightarrow 0$ . Recall that  $\Pi_{\tilde{R}}$  becomes the Poincaré polynomial  $\Pi_R$  under the latter limit;  $\Pi_R$  appears virtually in all constructions of the  $p$ -adic theory of spherical functions.

Opdam used the differential-*trigonometric* theory in [O1]; the rational theory alone appeared insufficient for (7.20). Its direct justification is known only in the  $A$ -case (Dunkl, Hanlon).

Another approach (originated in [O2]) is based on paper [GU] devoted to the semisimplicity of the (non-affine) Hecke algebras. It works for arbitrary finite Coxeter groups, that is beyond the frameworks of our method.

Reversing, in a sense, [O2] (or using [GGOR]), we can now apply the affine exponents to *reprove* the formula from [GU] in the crystallographic case. Applications of the affine exponents are expected for the semisimplicity of the Ariki–Koike–Cherednik algebras and for similar algebras. It is related to the  $q \leftrightarrow q'$ -duality of  $\Pi_{\tilde{R}}$ , which will not be discussed here. □

**7.4. Non-simply-laced case.** Recall that we use the normalization  $\nu_{\text{sht}} = 1$  for  $\nu_\alpha = (\alpha, \alpha)/2$  and  $q_\alpha = q^{\nu_\alpha}$ . We will begin with the formula in the case of the root system  $B_n$ ;  $\epsilon_m$  are from the  $B$ -table of

[B],  $(\epsilon_l, \epsilon_m) = 2\delta_{lm}$ . Then

$$\begin{aligned} R_+ = & \{\text{sht} : \epsilon_m = \alpha_m + \dots + \alpha_n, m = 1, \dots, n\} \text{ and} \\ & \{\text{lng}_- : \epsilon_l - \epsilon_m = \alpha_l + \dots + \alpha_{m-1}\}, \text{ as } n \geq m > l \geq 1, \\ & \{\text{lng}_+ : \epsilon_l + \epsilon_m = \alpha_l + \dots + \alpha_{m-1} + 2(\alpha_m + \dots + \alpha_n)\}; \end{aligned}$$

$$(\epsilon_m^\vee, \rho_k) = k_{\text{sht}} + 2(n-m)k_{\text{lng}}, \quad \rho_1 = \rho_{\text{sht}} = \sum_{m=1}^n \epsilon_m/2,$$

$$((\epsilon_l - \epsilon_m)^\vee, \rho_k) = (m-l)k_{\text{lng}}, \quad \rho_2 = \rho_{\text{lng}} = \sum_{m=1}^n (n-m)\epsilon_m,$$

$$((\epsilon_l + \epsilon_m)^\vee, \rho_k) = k_{\text{sht}} + (2n-m-l)k_{\text{lng}}, \quad \rho_k = k_{\text{sht}}\rho_1 + k_{\text{lng}}\rho_2.$$

Let us separate the terms with  $j = 1$  constituting the part of (7.9) without  $q$ . Recall that the factor  $t^{-\mathbf{ae}} q^{-(\rho, \rho+2\rho_k)}$  is disregarded in the definition of  $\Pi_{\tilde{R}}$ :

$$\begin{aligned} \Pi_{\tilde{R}} & \stackrel{\text{def}}{=} \prod_{\alpha \in R_+} \left( (1 - q_\alpha^{k_\alpha + (\alpha^\vee, \rho + \rho_k)}) \prod_{j=1}^{(\alpha^\vee, \rho)} \frac{(1 - q_\alpha^{j-1+k_\alpha + (\alpha^\vee, \rho_k)})}{(1 - q_\alpha^{j-1 + (\alpha^\vee, \rho_k)})} \right) \\ (7.21) \quad & = \frac{Q'_1}{Q'_2} \prod_{m=0}^{n-1} \left( \frac{1 - q^{2(m+1)k_{\text{lng}}}}{1 - q^{2k_{\text{lng}}}} \right) \left( \frac{1 - q^{2(2mk_{\text{lng}} + k_{\text{sht}})}}{1 - q^{2mk_{\text{lng}} + k_{\text{sht}}}} \right). \end{aligned}$$

The numerator of the product part ( $t$ -pure part) of the latter formula is obviously divisible by the denominator. It is much less obvious for  $Q'_1/Q'_2$  (this follows of course from the construction in terms of the shift operator). Let  $\tilde{Q}_1$  and  $\tilde{Q}_2$  be the *reduced numerator and denominator* upon reducing the *coinciding* factors but without the divisions,  $\tilde{Q}'_{1,2}$  the sub-products where all  $t$ -pure binomials are removed. Then

$$\begin{aligned} (7.22) \quad \tilde{Q}'_1 & = (1 - q^{2k_{\text{sht}}+1}) \prod_{m=2}^n \prod_{j=1}^{m-1} (1 - q^{2mk_{\text{lng}}+2j}) \\ & \times \prod_{m \in M} \prod_{j=1,2,\dots}^{i_m + \sigma_m j < 2+2m} (1 - q^{2mk_{\text{lng}}+2k_{\text{sht}}+i_m + \sigma_m j}), \end{aligned}$$

$$M = \{1, 2, \dots, 2[n/2], 2[n/2] + 2, 2[n/2] + 4, \dots, 2n - 2\},$$

the  $j$ -step  $\sigma_m$  is 1 for  $m = 2, 4, \dots, 2[(n+1)/2] + 2$ , and 2 otherwise; the shift  $i_m$  is  $-1$  for odd  $m$  and zero otherwise.



Let us give the list of exponents in the binomials  $(1 - q^{\{\cdot\}})$  that appear in  $\tilde{Q}'_1$  for  $n = 3$ :

$$(7.23) \quad \begin{aligned} &(1 + 2k_{\text{sht}}); (2 + 4k_{\text{lng}}); (2 + 6k_{\text{lng}}), (4 + 6k_{\text{lng}}); \\ &(1 + 2k_{\text{lng}} + 2k_{\text{sht}}), (3 + 2k_{\text{lng}} + 2k_{\text{sht}}); \\ &(1 + 4k_{\text{lng}} + 2k_{\text{sht}}), (2 + 4k_{\text{lng}} + 2k_{\text{sht}}), \\ &(3 + 4k_{\text{lng}} + 2k_{\text{sht}}), (4 + 4k_{\text{lng}} + 2k_{\text{sht}}), (5 + 4k_{\text{lng}} + 2k_{\text{sht}}); \\ &(2 + 8k_{\text{lng}} + 2k_{\text{sht}}), (4 + 8k_{\text{lng}} + 2k_{\text{sht}}), \\ &(6 + 8k_{\text{lng}} + 2k_{\text{sht}}), (8 + 8k_{\text{lng}} + 2k_{\text{sht}}). \end{aligned}$$

The formula for the *reduced* denominator  $\tilde{Q}'_2$  reads as follows:

$$(7.24) \quad \tilde{Q}'_2 = \prod_{m=1}^{n-1} \prod_{j=1}^{2m} (1 - q^{k_{\text{sht}} + 2mk_{\text{lng}} + j}).$$

The number of terms here is  $n(n-1)$ .

In the case of  $n = 3$ , the  $q$ -exponents of  $\tilde{Q}'_2$  are

$$(7.25) \quad \begin{aligned} &(1 + 2k_{\text{lng}} + k_{\text{sht}}), (2 + 2k_{\text{lng}} + k_{\text{sht}}); (1 + 4k_{\text{lng}} + k_{\text{sht}}), \\ &(2 + 4k_{\text{lng}} + k_{\text{sht}}), (3 + 4k_{\text{lng}} + k_{\text{sht}}), (4 + 4k_{\text{lng}} + k_{\text{sht}}). \end{aligned}$$

The number of factors in  $\tilde{Q}'_2$  is always smaller than in  $\tilde{Q}'_1$  by the number of positive roots  $|R_+|$  (9 in the case of  $B_3$ ).

## 8. AFFINE EXPONENTS

Continuing the previous section, we come to a general notion of the **affine exponents** that are the exponents of the terms  $(1 - q^{\{\cdot\}})$  in the *reduced denominator*  $\tilde{Q}_1$  and *reduced numerator*  $\tilde{Q}_2$  of  $\Pi_{\tilde{R}}$  from (7.21). Affine exponents can be *positive* (from the numerator) or *negative* (from the denominator); (7.23) and (7.25) give such exponents without the pure  $k$ -terms for  $\tilde{B}_3$ . The *reduction* is simple removing coinciding terms in the numerator and denominator without performing any actual divisions.

In contrast to the *classical exponents* (Coxeter exponents), the reduced denominators  $\tilde{Q}_2$  can be nontrivial if  $R$  is not simply-laced; in the simply-laced case, only the factors  $(1 - t)$  appear in  $\tilde{Q}_2$ .

Always, as in (7.15), the numerator  $\tilde{Q}_1$  is divisible by the denominator  $\tilde{Q}_2$ , which is not quite simple to justify without the approach based on the shift operator (in the non-simply-laced case). As in the classical case, the total divisibility guaranties that the coefficients of the  $q, t$ -expansion of  $\Pi_{\tilde{R}}$  in terms of the  $q, t$ -powers  $q_\nu^a t_{\text{sht}}^b t_{\text{lng}}^c$  for  $a, b, c \in \mathbb{Z}_+$  will be from  $\mathbb{Z}$ . Indeed, the final formula will be a product of terms in the form  $(1 - A^m)/(1 - A)$ , where  $A$  is a  $q, t$ -power, and also terms  $(1 - A)$  coming from the rational exponents. The latter terms, generally, result in negative coefficients; the coefficients would be all from  $\mathbb{Z}_+$  if the rational exponents were disregarded. The coefficients of the classical product  $\Pi_R$  are from  $\mathbb{Z}_+$ .

**8.1. Rational exponents.** In this paper, **rational exponents** play an important role. Their list is obtained from the list of affine exponents *in the numerator*  $\tilde{Q}_1$  by removing the binomials that are divisible by some binomials in the denominator  $\tilde{Q}_2$ . Only such exponents can lead to the  $k$ -roots of the product (7.21) provided that  $q$  is not a root of unity and imposing  $q^a \prod_\nu t_\nu^{b_\nu} = 1 \Rightarrow a + \sum_\nu \nu k_\nu b_\nu = 0$ . More formally, rational exponents are the affine *positive* (from the numerator) exponents, such that the corresponding binomials are not divisible by any binomials in the denominator. There is one-to-one correspondence between the non-rational exponents and the *negative* exponents (from the denominator), which makes this definition meaningful and also ensures that *the number of rational exponents is always  $|R_+|$* .

**Comment.** (i) Concerning “integrality” of the  $q$ -expansion of  $\Pi_{\tilde{R}}$ , we would like to discuss the formula for the graded multiplicities of the adjoint representation  $\mathfrak{g}$  of the simple Lie algebra  $\mathfrak{g}$  associated with  $R$  in the exterior algebra  $\Lambda \mathfrak{g}$ .

It was conjectured by Joseph in the simply-laced case that

$$(8.26) \quad \sum_{m \geq 0} [\Lambda^m \mathfrak{g} : \mathfrak{g}] q^m = (1 + q^{-1}) \prod_{i=1}^{n-1} (1 + q^{2m_i+1}) \sum_{i=1}^n q^{2m_i},$$

a variant/generalization of a series of formulas on the structure of  $\Lambda \mathfrak{g}$  due to Kostant and others. Bazlov extended it to the non-simply-laced case (which is nontrivial) and proved it in [Ba] by using the  $Y$ -operators.

(ii) He calculates the “non-affine” part of the expansion of  $\mu$  and interprets a proper sum of the corresponding coefficients *for a special*

value of  $t$  as (8.26). See also [Ion], where the method is somewhat different and additional information can be found.

We do not expect [Ba],[Ion] to be directly connected with our affine (or rational) exponents. Recall that the main part of  $\Pi_{\tilde{R}}$  comes from the principle value  $P_{-\rho}(q^{-\rho_k})$ , the  $q, t$ -dimension of  $V_\rho$ ; the latter does not appear in these two papers. However, the coefficients of the polynomial  $P_{-\rho}$  are connected with the coefficients calculated in [Ba],[Ion]. It may be an indication that these papers can be extended and some variants of (8.26) may be used to interpret  $P_{-\rho}(q^{-\rho_k})$  and  $\Pi_{\tilde{R}}$ .

(iii) Some of the *rational exponents* appear in [Ba] and [Ion]. It is not surprising because the  $k$ -zeros of rational exponents are the *singular  $k$ -parameters*, the values of  $k$  when  $\mathcal{V}$  has a nontrivial radical *Rad* of the evaluation pairing  $\{, \}$ . To be exact, such zeros give the cases when the  $t$ -discriminant  $\mathcal{X}^t$ , the  $t$ -anti-invariant for the non-affine Hecke subalgebra of  $\mathcal{H}^b$  from 7.1, belongs to *Rad* (which makes it nonzero). If  $k_\nu$  are replaced by the sets  $k_\nu + \mathbb{Z}_+$ , then the zeros of affine exponents describe *all* singular  $k$ .

This interpretation of the rational exponents is in the case when all multiplicative relations among  $q, t_\nu$  come from the  $\mathbb{Z}$ -relations among  $1, \nu k_\nu$ ; generally, the *affine exponents* must be used, say,  $t$  can be  $\zeta q^{s/r}$  for  $\zeta = \sqrt[r]{1} \neq 1$ .

Therefore the rational or affine exponents are inevitable in formulas that somehow require the non-degeneracy of the Macdonald polynomials and related structures.  $\square$

In the simply-laced case, as we know, the *rational exponents* are all positive *affine exponents* (from the numerator) with nonzero integer components  $j$ . They are  $\{j + k(m_i + 1), 1 \leq j \leq m_i\}$ . For  $\tilde{B}_n, \tilde{C}_n, \tilde{F}_4, \tilde{G}_2$ , the lists are given in the following theorem.

Due to their definition, the product of the *affine rational exponents* (as they are, without  $1 - q^{\{\cdot\}}$ ) is proportional to the limit of (7.21) as  $q \rightarrow 1$ . So this product describes  $(\prod_{\alpha \in R_+} y_\alpha)((\prod_{\alpha \in R_+} x_\alpha))$ , which generalizes Corollary 7.4 to arbitrary, possibly non-simply-laced, root systems.

The products of affine rational exponents in the non-simply-laced cases are proportional to those from [DJO], Theorem 4.11. Recall that we process the product (7.21) combinatorially, by canceling coinciding factors as one does with the classical  $\Pi_R$ . The “non-divisible” binomials (from the numerator) lead to *affine rational exponents*.

**Comment.** (i) The multiplicities of the rational exponents in the limit are the only lost information if the rational exponents are defined entirely within the rational theory, i.e., without the above approach via the rational affine exponents in the  $q, t$ -theory. The nontrivial multiplicities create technical difficulties for Opdam’s approach. Due to the presence of  $1 - q^{\{\cdot\}}$  in the affine formulas, proportional rational exponents come from *distinct* factors in  $\Pi_{\tilde{R}}$  in our approach (unless for  $D_{\text{even}}$ !). The cases of nontrivial multiplicities in the limit are as follows.

For  $\tilde{B}_n, n \geq 4$ , such pairs are  $\{mk_{\text{lng}} + j, lmk_{\text{lng}} + lj\}$  as  $2 \leq lm \leq n, j/m \notin \mathbb{Z}$ . For  $\tilde{C}_n$ ,  $k_{\text{lng}}$  must be replaced here by  $k_{\text{sht}}$ . Also, the pairs of the rational exponents  $\{1+2k_{\text{lng}}+2k_{\text{sht}}, 3+6k_{\text{lng}}+6k_{\text{sht}}\}, \{3+2k_{\text{lng}}+2k_{\text{sht}}, 9+6k_{\text{lng}}+6k_{\text{sht}}\}$  for  $\tilde{F}_4$  from (8.30) cannot be distinguished under the rational limit.

These are the only cases when the multiplicities occur in the rational limit for generic  $k_{\text{sht}}, k_{\text{lng}}$  (in the non-simply-laced case). If  $k_{\text{sht}} = \nu_{\text{lng}} k_{\text{lng}}$  (for instance, for  $A, D, E$ ), then there are many multiple exponents in the rational limit.

We stick to the “combinatorial” (via  $q, t$ ) definition of the affine rational exponents in this paper.

(ii) Technically, the reduction of coinciding factors is somewhat simpler among the affine rational exponents, but the case of “divisible” (non-rational) exponents is not very different; see below. Note that the duality between the rational exponents of  $\tilde{B}$  and  $\tilde{C}$  from the theorem does not hold for the affine exponents (its certain affine counterpart exists but is of more sophisticated nature).

However, Theorem 8.2 below shows that the strict duality holds for *another* affine extension of  $R$ , namely, for  $R \times \mathbb{Z} = \{[\alpha, j]\}$ .

Under the rational limit, the rational  $B \leftrightarrow C$ -duality simply reflects the fact that the rational Dunkl operators are  $k_{\text{sht}} \leftrightarrow k_{\text{lng}}$ -coinciding for  $B$  and  $C$ . Indeed, in the rational case, the construction of these operators does not depend on the choice of the affine extension of  $R$ .

(iii) We define in this paper the affine roots in the form  $[\alpha, \nu_{\alpha} j]$  (with the  $\nu_{\alpha}$ -factors) and introduce DAHA using the *coinciding* affine systems  $\{\tilde{R}, \tilde{R}\}$  for both,  $X$  and  $Y$ . This makes  $\mathcal{H}$  invariant under the action of the automorphisms  $\tau_{\pm}, \sigma$  and simplifies other considerations, like the theory of the evaluation pairing  $\{\cdot, \cdot\}$ . Almost all results of this paper can be transferred to the case of the affine root system

$\widehat{R} \stackrel{\text{def}}{=} R \times \mathbb{Z}$  and, moreover, the for combination  $\widetilde{R}, \widehat{R}$  taken for  $X$  and for  $Y$  in the definition of  $\mathcal{H}$  and related structures. Note that DAHA and the  $q, t$ -shift-operator from [C4]) were defined for the combination  $\{\widehat{R}, \widetilde{R}\}$  of affine extensions. Such choice is exactly needed for the affine duality discussed below.

There is also flexibility with choosing the lattices  $Q \subset B \subset P$  for  $X$  and the lattice between  $Q^\vee$  and  $P^\vee$  for the generators  $Y$ . However, the choice of the lattices does not affect the definition of the shift-operator.  $\square$

**Theorem 8.1.** (i) *The number of rational exponents is always  $|R_+|$  and they are all simple (unless for  $D_{\text{even}}$ ). In the  $\widetilde{B}_n$ -case, the list of the corresponding powers in the binomials  $(1 - q^{\{\cdot\}})$  is*

$$(8.27) \quad \{2k_{\text{sht}} + 1\}, \{2mk_{\text{lng}} + 2j, 2 \leq m \leq n, 0 < j < m\}, \\ \{2k_{\text{sht}} + 2mk_{\text{lng}} + 2j + 1, 1 \leq m < n, 0 \leq j \leq m\}.$$

*Up to proportionality, the rational exponents in the  $\widetilde{C}_n$ -case are obtained from (8.27) by the transposition  $k_{\text{lng}} \leftrightarrow k_{\text{sht}}$ . Explicitly:*

$$(8.28) \quad \{2k_{\text{lng}} + 1\}, \{mk_{\text{sht}} + j, 2 \leq m \leq n, 0 < j < m\}, \\ \{2k_{\text{lng}} + 2mk_{\text{sht}} + 2j + 1, 1 \leq m < n, 0 \leq j \leq m\}.$$

(ii) *In the case of  $\widetilde{G}_2$ , the list is:*

$$(8.29) \quad \{(1 + 2k_{\text{lng}}), (1 + 2k_{\text{sht}}), (1 + 3k_{\text{lng}} + 3k_{\text{sht}}), \\ (2 + 3k_{\text{lng}} + 3k_{\text{sht}}), (4 + 3k_{\text{lng}} + 3k_{\text{sht}}), (5 + 3k_{\text{lng}} + 3k_{\text{sht}})\}.$$

(iii) *In the case of  $\widetilde{F}_4$ , the list is:*

$$(8.30) \quad \{(2 + 4k_{\text{lng}}), (2 + 6k_{\text{lng}}), (4 + 6k_{\text{lng}}), \\ (1 + 2k_{\text{sht}}), (1 + 3k_{\text{sht}}), (2 + 3k_{\text{sht}}), \\ (1 + 2k_{\text{lng}} + 2k_{\text{sht}}), (3 + 2k_{\text{lng}} + 2k_{\text{sht}}), \\ (2 + 8k_{\text{lng}} + 4k_{\text{sht}}), (6 + 8k_{\text{lng}} + 4k_{\text{sht}}), (10 + 8k_{\text{lng}} + 4k_{\text{sht}}), \\ (1 + 2k_{\text{lng}} + 4k_{\text{sht}}), (3 + 2k_{\text{lng}} + 4k_{\text{sht}}), (5 + 2k_{\text{lng}} + 4k_{\text{sht}}), \\ (1 + 4k_{\text{lng}} + 4k_{\text{sht}}), (3 + 4k_{\text{lng}} + 4k_{\text{sht}}), \\ (5 + 4k_{\text{lng}} + 4k_{\text{sht}}), (7 + 4k_{\text{lng}} + 4k_{\text{sht}}), \\ (1 + 6k_{\text{lng}} + 6k_{\text{sht}}), (3 + 6k_{\text{lng}} + 6k_{\text{sht}}), (5 + 6k_{\text{lng}} + 6k_{\text{sht}}), \\ (7 + 6k_{\text{lng}} + 6k_{\text{sht}}), (9 + 6k_{\text{lng}} + 6k_{\text{sht}}), (11 + 6k_{\text{lng}} + 6k_{\text{sht}})\}.$$

□

8.2. **The case of  $\tilde{C}_n$ .** Now the inner product is  $(\epsilon_l, \epsilon_m) = \delta_{lm}$  and

$$R_+ = \{ 2\epsilon_m, m = 1, \dots, n, \text{ and } \epsilon_l \pm \epsilon_m, n \geq m > l \geq 1 \},$$

$$((2\epsilon_m)^\vee, \rho_k) = k_{\text{lng}} + (n - m)k_{\text{sht}}, \quad \rho_2 = \rho_{\text{lng}} = \sum_{m=1}^n \epsilon_m,$$

$$((\epsilon_l - \epsilon_m)^\vee, \rho_k) = (m - l)k_{\text{sht}}, \quad \rho_1 = \rho_{\text{sht}} = \sum_{m=1}^n (n - m)\epsilon_m,$$

$$((\epsilon_l + \epsilon_m)^\vee, \rho_k) = 2k_{\text{lng}} + (2n - m - l)k_{\text{sht}}, \quad \rho_k = k_{\text{sht}}\rho_1 + k_{\text{lng}}\rho_2.$$

The complete formula for the product  $\Pi_{\tilde{C}_n}$  reads as:

$$(8.31) \quad \prod_{\alpha \in R_+} \left( (1 - q_\alpha^{k_\alpha + (\alpha^\vee, \rho + \rho_k)}) \prod_{j=1}^{(\alpha^\vee, \rho)} \frac{(1 - q_\alpha^{j-1+k_\alpha + (\alpha^\vee, \rho_k)})}{(1 - q_\alpha^{j-1+(\alpha^\vee, \rho_k)})} \right) \\ = \frac{\prod_{m=2}^n \prod_{j=0}^{m-1} (1 - q^{mk_{\text{sht}}+j})}{(1 - q^{k_{\text{sht}}})^{n-1} \prod_{m=0}^{n-1} \prod_{\substack{\text{even } j \\ 0 \leq j \leq m+1}} (1 - q^{2k_{\text{lng}}+mk_{\text{sht}}+j})} \\ \times \prod_{m=\lfloor \frac{n+1}{2} \rfloor}^{n-1} \prod_{\substack{j \bmod 2=1 \\ 1 \leq j \leq 2m+1}} (1 - q^{2k_{\text{lng}}+2mk_{\text{sht}}+j}) \\ \times \prod_{m=0}^{n-1} \prod_{\substack{j \bmod 2=0 \\ 0 \leq j \leq 2m+2}} (1 - q^{4k_{\text{lng}}+2mk_{\text{sht}}+j}).$$

We did not separate the terms that come from the Poincaré polynomial  $\Pi_R$  and the terms that contain  $q$ ; see (7.17). Generally, this product looks somewhat simpler than the one for  $\tilde{B}_n$ . We do not have any statements about their connection.

Let us give the formula for the number  $N_-(C_n)$  of *negative affine exponents* (i.e., in the denominator):

$$N_-(C_n) = n - 2 + \left\lfloor \frac{n+1}{2} \right\rfloor \left\lfloor \frac{n+3}{2} \right\rfloor / 2 + \left\lfloor \frac{n+2}{2} \right\rfloor^2.$$

Respectively,  $N_+(C_n) = |R_+| + N_-(C_n) = n^2 + N_-(C_n)$ . These formulas for  $B_n$  are very different:  $N_-(B_n) = n^2 + n + 1$ ; see (7.24).

**8.3. The case of  $\tilde{G}_2$ .** We will give the complete formula for the affine exponents in the  $\tilde{G}_2$ -case. The following is the list of the  $k$ -heights  $(\alpha^\vee, \rho_k)$  in the notation of [B]:

$$\begin{aligned}
 &\text{short } \alpha_1, & (\alpha^\vee, \rho_k) &= k_{\text{sht}}, \\
 &\text{short } \alpha_1 + \alpha_2, & (\alpha^\vee, \rho_k) &= 3k_{\text{lng}} + k_{\text{sht}}, \\
 &\text{short } 2\alpha_1 + \alpha_2, & (\alpha^\vee, \rho_k) &= 3k_{\text{lng}} + 2k_{\text{sht}}, \\
 &\text{long } \alpha_2, & (\alpha^\vee, \rho_k) &= k_{\text{lng}}, \\
 &\text{long } 3\alpha_1 + \alpha_2, & (\alpha^\vee, \rho_k) &= k_{\text{lng}} + k_{\text{sht}}, \\
 &\text{long } 3\alpha_1 + 2\alpha_2, & (\alpha^\vee, \rho_k) &= 2k_{\text{lng}} + k_{\text{sht}}, \\
 (8.32) \quad &\rho_k &= (2\alpha_1 + \alpha_2)k_{\text{sht}} + (3\alpha_1 + 2\alpha_2)k_{\text{lng}}.
 \end{aligned}$$

Recall that the normalization is  $(\alpha_1, \alpha_1) = 2$ .

The 12 *affine exponents* in the numerator are:

$$\begin{aligned}
 (8.33) \quad &(6k_{\text{lng}}), (3 + 6k_{\text{lng}}), (2k_{\text{sht}}), (1 + 2k_{\text{sht}}), \\
 &(1 + 3k_{\text{lng}} + 3k_{\text{sht}}), (2 + 3k_{\text{lng}} + 3k_{\text{sht}}), (4 + 3k_{\text{lng}} + 3k_{\text{sht}}), \\
 &(5 + 3k_{\text{lng}} + 3k_{\text{sht}}), (9k_{\text{lng}} + 3k_{\text{sht}}), (3 + 9k_{\text{lng}} + 3k_{\text{sht}}), \\
 &(6 + 9k_{\text{lng}} + 3k_{\text{sht}}), (9 + 9k_{\text{lng}} + 3k_{\text{sht}}).
 \end{aligned}$$

The 6 *affine exponents* in the denominator are:

$$\begin{aligned}
 (8.34) \quad &\{(3k_{\text{lng}}), (k_{\text{sht}}), (3k_{\text{lng}} + k_{\text{sht}}), (1 + 3k_{\text{lng}} + k_{\text{sht}}), \\
 &(2 + 3k_{\text{lng}} + k_{\text{sht}}), (3 + 3k_{\text{lng}} + k_{\text{sht}})\}.
 \end{aligned}$$

**8.4. The case of  $\tilde{F}_4$ .** In the notation of [B], the roots are

$$\begin{aligned}
 &\text{lng: } 1000, 0100, 1100, 0120, 1120, 1220, \\
 &\quad 0122, 1122, 1222, 1242, 1342, 2342, \\
 &\text{sht: } 0010, 0001, 0110, 0011, 0111, 1110, \\
 &\quad 1111, 0121, 1121, 1221, 1231, 1232; \\
 (8.35) \quad &(\alpha^\vee, \rho_k) = l_{\nu, \text{lng}}(a + b)k_{\text{lng}} + l_{\nu, \text{sht}}^{-1}(c + d)k_{\text{sht}}, \\
 &\nu = \nu_{abcd}, \quad l_{\text{lng}, \text{sht}} = 2 = l_{\text{sht}, \text{lng}}, \quad l_{\text{lng}, \text{lng}} = 1 = l_{\text{sht}, \text{sht}}.
 \end{aligned}$$

Let us give the list of the *corresponding*  $(\alpha^\vee, \rho_k)$  :

$$\begin{aligned} \underline{\text{lng}}: & k_{\text{lng}}, k_{\text{lng}}, 2k_{\text{lng}}, k_{\text{lng}} + k_{\text{sht}}, 2k_{\text{lng}} + k_{\text{sht}}, 3k_{\text{lng}} + k_{\text{sht}}, k_{\text{lng}} + 2k_{\text{sht}}, \\ & 2k_{\text{lng}} + 2k_{\text{sht}}, 3k_{\text{lng}} + 2k_{\text{sht}}, 3k_{\text{lng}} + 3k_{\text{sht}}, 4k_{\text{lng}} + 3k_{\text{sht}}, 5k_{\text{lng}} + 3k_{\text{sht}}; \\ \underline{\text{sht}}: & k_{\text{sht}}, k_{\text{sht}}, 2k_{\text{lng}} + k_{\text{sht}}, 2k_{\text{sht}}, 2k_{\text{lng}} + 2k_{\text{sht}}, 4k_{\text{lng}} + k_{\text{sht}}, 4k_{\text{lng}} + 2k_{\text{sht}}, \\ & 2k_{\text{lng}} + 3k_{\text{sht}}, 4k_{\text{lng}} + 3k_{\text{sht}}, 6k_{\text{lng}} + 3k_{\text{sht}}, 6k_{\text{lng}} + 4k_{\text{sht}}, 6k_{\text{lng}} + 5k_{\text{sht}}. \end{aligned}$$

The 47 *affine exponents* in the numerator:

$$\begin{aligned} (8.36) \quad & (4k_{\text{lng}}), (2 + 4k_{\text{lng}}), (6k_{\text{lng}}), (2 + 6k_{\text{lng}}), (4 + 6k_{\text{lng}}), \\ & (2k_{\text{sht}}), (1 + 2k_{\text{sht}}), (3k_{\text{sht}}), (1 + 3k_{\text{sht}}), (2 + 3k_{\text{sht}}), \\ & (1 + 2k_{\text{lng}} + 4k_{\text{sht}}), (3 + 2k_{\text{lng}} + 4k_{\text{sht}}), (5 + 2k_{\text{lng}} + 4k_{\text{sht}}), \\ & (8k_{\text{lng}} + 2k_{\text{sht}}), (2 + 8k_{\text{lng}} + 2k_{\text{sht}}), \dots, (8 + 8k_{\text{lng}} + 2k_{\text{sht}}), \\ & (4k_{\text{lng}} + 4k_{\text{sht}}), (1 + 4k_{\text{lng}} + 4k_{\text{sht}}), \dots, (7 + 4k_{\text{lng}} + 4k_{\text{sht}}), \\ & (8k_{\text{lng}} + 4k_{\text{sht}}), (2 + 8k_{\text{lng}} + 4k_{\text{sht}}), \dots, (10 + 8k_{\text{lng}} + 4k_{\text{sht}}), \\ & (1 + 6k_{\text{lng}} + 6k_{\text{sht}}), (3 + 6k_{\text{lng}} + 6k_{\text{sht}}), \dots, (11 + 6k_{\text{lng}} + 6k_{\text{sht}}), \\ & (12k_{\text{lng}} + 6k_{\text{sht}}), (2 + 12k_{\text{lng}} + 6k_{\text{sht}}), \dots, (16 + 12k_{\text{lng}} + 6k_{\text{sht}}). \end{aligned}$$

Note that the integer step here is 1 for  $2k_{\text{sht}}, 3k_{\text{sht}}$  and  $(4k_{\text{lng}} + 4k_{\text{sht}})$  or 2 otherwise.

The 23 *affine exponents* in the denominator:

$$\begin{aligned} (8.37) \quad & (2k_{\text{lng}}), (2k_{\text{lng}}), (k_{\text{sht}}), (k_{\text{sht}}), (2k_{\text{lng}} + k_{\text{sht}}), (1 + 2k_{\text{lng}} + k_{\text{sht}}), \\ & (2 + 2k_{\text{lng}} + k_{\text{sht}}), (2k_{\text{lng}} + 2k_{\text{sht}}), (2 + 2k_{\text{lng}} + 2k_{\text{sht}}), \\ & (4k_{\text{lng}} + k_{\text{sht}}), (1 + 4k_{\text{lng}} + k_{\text{sht}}), \dots, (4 + 4k_{\text{lng}} + k_{\text{sht}}), \\ & (6k_{\text{lng}} + 3k_{\text{sht}}), (1 + 6k_{\text{lng}} + 3k_{\text{sht}}), \dots, (8 + 6k_{\text{lng}} + 3k_{\text{sht}}). \end{aligned}$$

**8.5. Affine duality.** Without going into detail let us formulate the theorem about the variant of  $\Pi_{\tilde{R}}$  for *another choice of the affine extension* of  $R$ ; let  $\hat{R} \stackrel{\text{def}}{=} \{[\alpha \in R, j \in \mathbb{Z}]\}$ . For such affine system, the factors  $\nu_\alpha$  do not appear in the integer components of the affine roots and, respectively, we will not need  $q_\nu$  in the formula for  $\Pi_{\hat{R}}$ .

There is no change in the simply-laced case, so only the non-simply-laced systems are sufficient to consider. We come to the following variant of the previous construction.



The corresponding DAHA and the shift-operator will be now for the pair of affine root system  $\{\widehat{R}, \widetilde{R}\}$  used respectively for  $X, Y$  as in [C4]. The connection with the radical of  $\mathcal{V}$  is very much the same as it is for  $\{\widetilde{R}, \widetilde{R}\}$ , which is the only case considered in this paper (Theorem 11.8). The duality below is a variant of the Langlands duality and is directly related to the Fourier transform for DAHA of type  $\{\widehat{R}, \widetilde{R}\}$ .

For this combination of affine extensions, the composition  $\mathcal{Y} \circ \mathcal{X}$  is invariant with respect to the DAHA–Fourier transform (treated as an abstract anti-involution); this leads to the required duality.

The rational DAHA do not depend on the particular choice of the pair of affine root systems and *rational exponents* remain the same for any such choices *up to proportionality*. The corresponding binomials in  $\Pi_{\widehat{R}}$  are  $(1 - q^{\{rat\ exp\}})$ ; upon the substitutions  $t_\alpha = q^{k_\alpha}$ , they can be different from those in  $\Pi_{\widetilde{R}}$ , where we set  $t_\alpha = q^{\nu_\alpha k_\alpha}$ . Note that the binomials in terms of  $t_{\text{lng}}, t_{\text{sht}}$  only (without nontrivial powers of  $q$ ) remain the same as for  $\Pi_{\widetilde{R}}$ ; the Poincaré polynomial is unchanged.

**Theorem 8.2.** (i) In the cases  $\widehat{B}_n, \widehat{C}_n, \widehat{F}_4, \widehat{G}_2$ , the product

$$(8.38) \quad \Pi_{\widehat{R}} \stackrel{\text{def}}{=} \prod_{\alpha \in R_+} \left( (1 - q^{k_\alpha + (\alpha^\vee, \rho + \rho_k)}) \prod_{j=1}^{(\alpha^\vee, \rho)} \frac{(1 - q^{j-1+k_\alpha + (\alpha^\vee, \rho_k)})}{(1 - q^{j-1+(\alpha^\vee, \rho_k)})} \right)$$

is a regular function with the corresponding rational exponents that are proportional to those from Theorem 8.1 for  $\Pi_{\widetilde{R}}$ . The affine duality holds:

$$\Pi_{\widehat{R}}(q, k_{\text{lng}}, k_{\text{sht}}) = \Pi_{\widehat{R}^\vee}(q, k_{\text{sht}}, k_{\text{lng}}).$$

(ii) The  $\Pi_{\widehat{R}}$ -product for  $\widehat{B}_n$  equals

$$(8.39) \quad \frac{\prod_{m=2}^n \prod_{j=0}^{m-1} (1 - q^{mk_{\text{lng}} + j}) \prod_{m=0}^{n-1} \prod_{j=0}^{2m+1} (1 - q^{2k_{\text{sht}} + 2mk_{\text{lng}} + j})}{(1 - q^{k_{\text{lng}}})^{n-1} \prod_{m=0}^{n-1} \prod_{j=0}^m (1 - q^{k_{\text{sht}} + mk_{\text{lng}} + j})}$$

with  $\frac{n^2+3n-2}{2}$  negative affine exponents. The proportional pairs of rational exponents here are  $\{mk_{\text{lng}} + j, lm k_{\text{lng}} + lj\}$  as  $j/m \notin \mathbb{Z}$  and  $lm < n, m > 1$ .

(iii) The affine exponents for  $\widehat{G}_2$  are

$$(8.40) \quad \begin{aligned} &\text{positive: } (2k_{\text{lng}}), (2k_{\text{sht}}), (1 + 2k_{\text{lng}}), (1 + 2k_{\text{sht}}), \\ &\quad \{ (j + 3k_{\text{lng}} + 3k_{\text{sht}}), j = 0, 1, 2, 3, 4, 5 \}, \\ &\text{negative: } (k_{\text{lng}}), (k_{\text{sht}}), (k_{\text{lng}} + k_{\text{sht}}), (1 + k_{\text{lng}} + k_{\text{sht}}), \end{aligned}$$

with  $10/4$  positive/negative exponents. All rational exponents are pairwise not proportional.

(iv) The affine exponents for  $\widehat{F}_4$  are  $(lk_{\text{lng}} + sk_{\text{sh}} + j)$ , where  $j = 0, 1, \dots, l + s - 1$  and

$$(8.41) \quad \begin{aligned} \text{positive: } [ls] &\in \{[20], [02], [30], [03], [24], [42], [44], [66]\}, \\ \text{negative: } [ls] &\in \{[10], [01], [10], [01], [12], [21], [11], [33]\}, \end{aligned}$$

with  $42/18$  positive/negative exponents. The proportional pairs of rational exponents are  $\{[44] + 2, [66] + 3\}$  and  $\{[44] + 6, [66] + 9\}$ .  $\square$

## 9. THE CHAIN OF INTERTWINERS

We are going to develop the technique of intertwiners aiming at decomposing the polynomial representation  $\mathcal{V}$  when the action of the  $Y$ -operators is non-semisimple. We will start with some basic properties of the generalized eigenvectors.

Recall that  $c_{\sharp} = c - u_c^{-1}(\rho_k) = \pi_c((- \rho_k))$  for  $c \in B$ , where the affine action  $((\cdot))$  from (1.29) is used. For instance,  $-0_{\sharp} = \rho_k$ . See (6.10) and (6.9).

**9.1. Generalized eigenvectors.** This section is for arbitrary nonzero  $q, t$  including the case when  $q$  are roots of unity. Almost all statements will require the constraint  $t_{\nu} \neq 1$ . For instance, we will constantly use that

$$q_{\alpha}^{(\tilde{\alpha}^{\vee}, c_{-} + d) - (\alpha^{\vee}, \rho_k)} \neq t_{\alpha}^{\pm 1} \text{ if } q_{\alpha}^{(\tilde{\alpha}^{\vee}, c_{-} + d) - (\alpha^{\vee}, \rho_k)} = 1.$$

Sometimes  $t_{\nu} \neq \pm 1$  will be needed; later it will be imposed permanently. There will be also quite a few claims that hold for generic  $q$  only; this condition will be stated explicitly.

The space of the **generalized eigenvectors**  $\mathcal{V}^{\infty}(\xi) \subset \mathcal{V}$  corresponding to a given weight  $\xi = -c_{\sharp}$  is as follows:

$$(9.1) \quad \begin{aligned} \mathcal{V}^s(\xi) &\stackrel{\text{def}}{=} \{v \in \mathcal{V} \mid (Y_a - q^{(a, \xi)})^s(v) = 0\}, \\ \mathcal{V}(\xi) &= \mathcal{V}^1(\xi), \quad \mathcal{V}^{\infty}(\xi) = \cup_{s>0} \mathcal{V}^s(\xi), \end{aligned}$$

for all  $a \in B$ .

The symbols  $q^{\xi}$  and the weights  $\xi$  are always identified if the corresponding characters  $a \mapsto q^{(a, \xi)}$  coincide for  $a \in B$ .

The spaces  $\mathcal{V}(-c_{\sharp})^{\infty}$  are finite dimensional for generic  $q$  and infinite dimensional when  $q$  is a root of unity. The dimension of the space

$\mathcal{V}(-c_{\sharp})^{\infty}$  equals the number of  $b$  satisfying (9.1). Given an *arbitrary one-parametric deformation*  $q^l, \{t_{\nu}^l\}$  of  $q, t$  which makes the polynomial representation semisimple, the space  $\mathcal{V}(-c_{\sharp})^{\infty}$  is the limit  $\{q^l \rightarrow q, t^l \rightarrow t\}$  of the linear space that is the direct sum

$$\oplus \mathbb{Q}_{q^l, t^l} E_b^l \text{ such that } q^{-c_{\sharp}} = q^{-b_{\sharp}}$$

for the Macdonald polynomials  $E_b^l$  defined for  $q^l, t^l$ .

The limit is defined in a sense of vector bundles over curves, to be exact, over a small one-dimensional disc with the center at  $q, t$ . We will call this limit *flat limit*. The corresponding space is the linear span of the limits of all linear combinations of  $E_b^l$  divided by proper powers of the deformation parameter. Obviously,  $\mathcal{V} = \oplus_c \mathcal{V}(-c_{\sharp})^{\infty}$  for pairwise different  $q^{-c_{\sharp}}$  (i.e., if the characters  $q^{-(a, c_{\sharp})}$  are different for  $a \in B$ ).

The space  $\mathcal{V}(-c_{\sharp})$  always contains at least one Macdonald polynomial  $E_c$ . Indeed, one can take  $c = c^{\circ}$  assuming that

$$(9.2) \quad q^{-c_{\sharp}} = q^{-c_{\sharp}^{\circ}} \text{ and } q^{-b_{\sharp}} \neq q^{-c_{\sharp}^{\circ}} \text{ for all } b \text{ such that } B \ni b \succ c^{\circ}.$$

Under this assumption, there is a unique  $Y$ -eigenvector of weight  $-c_{\sharp}$  in the space  $\oplus_{b \succeq c^{\circ}} \mathbb{Q}_{q, t} X_b$ ; it is proportional to  $E_{c^{\circ}}$ . We will call such  $c^{\circ}$  **primary** elements; they exist for any  $c$ .

We are going to apply the simple intertwiners  $P_r, \Psi_i$  from (6.15) to the spaces  $\mathcal{V}(-c_{\sharp})$  following (6.21). The  $P_r$  are always well defined and invertible. The intertwiner  $\Psi_i$  is not well defined in  $\mathcal{V}(-c_{\sharp})$  if and only if  $\Psi_i^c$  from (6.19), an element in the non-affine Hecke algebra  $\mathbf{H}$ , is infinity. The latter occurs exactly when

$$(9.3) \quad q_{\alpha}^{(\tilde{\alpha}^{\vee}, c_{-} + d) - (\alpha^{\vee}, \rho_k)} = 1,$$

where we set

$$\tilde{\alpha} = u_c(\alpha_i) \text{ and } \alpha = u_c(\alpha_i) \text{ for } i > 0, \alpha = u_c(-\vartheta) \text{ for } i = 0.$$

Sometimes it will be convenient to renormalize the  $\Psi$ -intertwiners as follows:

$$(9.4) \quad \Psi_i^{\diamond} = \Psi_i \cdot (Y_{\alpha_i}^{-1} - 1), \quad P_r^{\diamond} = P_r, \quad 0 \leq i \leq n, \quad r \in O.$$

Given  $\hat{w} \in \widehat{W}$ , the corresponding  $\Psi^{\diamond}$ -intertwiner  $\Psi_{\hat{w}}^{\diamond}$  maps  $\mathcal{V}^s(-c_{\sharp})$  to  $\mathcal{V}^s(-b_{\sharp})$  for  $b = \hat{w}((c))$ . In contrast to  $\Psi_{\hat{w}}$ , the intertwiners  $\Psi_{\hat{w}}^{\diamond}$  are always well defined, although they can be non-invertible and identically zero in some  $\mathcal{V}(-c_{\sharp})$ .

**9.2. The  $\tilde{E}$ -polynomials.** Let  $\tilde{V}_0 = \mathbb{Q}_{q,t}1 \subset \mathcal{V}$ . Given a reduced decomposition  $\pi_c = \pi_r s_{i_l} \cdots s_{i_1}$  for  $c \in B$ , we define  $\tilde{V}_c$  by induction as follows.

- (a) Let  $\tilde{V}_b = \Psi_i(\tilde{V}_c) = \Psi_i^\circ(\tilde{V}_c)$  for  $b = s_i((c))$  unless  $\Psi_i^c \in \mathbf{H}$  is infinity.
- (b) If  $\Psi_i^c \in \mathbf{H}$  is infinity, i.e., (9.3) holds, then  $\tilde{V}_b \stackrel{\text{def}}{=} \tilde{V}_c + \tau_+(T_i)(\tilde{V}_c)$ .
- (c) For  $\pi_r \in \Pi^b = \{\pi_r, \omega_r \in B\}$ , let  $\tilde{V}_b \stackrel{\text{def}}{=} P_r(\tilde{V}_c)$ , where  $b = \pi_r((c))$ .

Note that  $(\alpha_i, c + d) \neq 0$  if and only if  $s_i \pi_c$  is represented in the form  $\pi_b$ . This always holds for (b). Indeed, using  $\tilde{\alpha} = u_c(\alpha_i)$ , one gets  $(\alpha_i, c + d) = 0 \Rightarrow (\tilde{\alpha}, c_- + d) = 0$ ; the latter is impossible due to (9.3).

Given a reduced decomposition,  $\pi_c = \pi_r s_{i_l} \cdots s_{i_1}$ , we set

$$(9.5) \quad \begin{aligned} \tilde{\Psi}_{i_p} &= \Psi_{i_p} \quad \text{if } q_\alpha^{(\tilde{\alpha}^\vee, b_- + d) - (\alpha^\vee, \rho_k)} \neq 1 \\ &\quad \text{for } b = s_{i_{p-1}} \cdots s_{i_1}((0)), \quad \tilde{\alpha} = u_b(\alpha_{i_p}), \end{aligned}$$

$$(9.6) \quad \tilde{\Psi}_{i_p} = \tau_+(T_{i_p}) \quad \text{if } q_\alpha^{(\tilde{\alpha}^\vee, b_- + d) - (\alpha^\vee, \rho_k)} = 1.$$

Equivalently, the elements  $s_{i_p}$  from (9.6) are **singular**, that is  $\tilde{\alpha}^p \in \tilde{R}^0$ ; see the definition of  $\tilde{R}^0$  in (9.7) below. The intertwiner  $\tilde{\Psi}_{i_p}$  will be called singular too.

Given  $c \in B$ , we define the **non-semisimple polynomial**  $\tilde{E}_c \in \tilde{V}_c$  to be proportional to

$$P_r \tilde{\Psi}_{i_l} \cdots \tilde{\Psi}_{i_p} \cdots \tilde{\Psi}_{i_1}(1).$$

Note that always  $\tilde{\Psi}_{i_p}(\tilde{E}_c) = 0$  if  $(\alpha_{i_p}^\vee, c + d) = 0$ , i.e., when  $s_{i_p} \pi_c \notin \pi_B$ . It may happen only for *non-singular*  $s_{i_p}$ , i.e., when (9.5) holds. In contrast to the semisimple case, the specializations  $\Psi_i^c$  are sufficient only as  $\dim \tilde{V}_c = 1$ . Generally, the whole  $\Psi_i$  in terms of  $Y$  from (6.15) have to be involved.

The definition of  $\tilde{E}_c$  can be naturally extended to *possibly non-reduced* decompositions of  $\pi_c$ . In this case, the notation will be  $\tilde{E}_c^\dagger$ ; we call these polynomials **non-semisimple non-reduced**.

We will mainly need  $\tilde{E}$ -polynomials and  $\tilde{V}$ -spaces for reduced decompositions. Note that these polynomials are defined so far only up to proportionality and *depend on the choice of the decomposition* of  $\pi_c$ . The  $\tilde{V}$ -spaces *depend on the particular choice of the reduced decomposition* too but their dependence can be controlled, as we will see later.

Given a decomposition of an *arbitrary*  $\widehat{w} \in \widehat{W}$ , one can introduce  $\widetilde{E}_c$  and  $\widetilde{E}_c^\dagger$  for  $c = \widehat{w}((0))$  in the same manner; this definition coincides with the previous one if  $\widehat{w} = \pi_c$  is taken.

Generalizing, this construction can be originated at a given Macdonald polynomial  $E_c$  instead of  $E_0 = 1$ . For instance, one may begin with *primary*  $c = c^\circ$  from (9.2), when  $q^{-a_\sharp} \neq q^{-c^\circ_\sharp}$  for all  $a \succ c^\circ$ . Given  $c$ , we define  $\widetilde{E}_b$  for a decomposition of  $\widehat{w}$  such that  $\widehat{w}((c)) = b$ . The same notation  $\widetilde{E}_b$  will be used we will explicitly mention the **chain origin** if necessary. The reduced decompositions of  $\widehat{w} = \pi_b \pi_c^{-1}$  will be mainly needed, i.e., those satisfying  $l(\widehat{w}) = l(\pi_b) - l(\pi_c)$ ; otherwise  $\dagger$  will be added.

We call the resulting sequence  $\{\widetilde{E}_b, \dots, \widetilde{E}_c = E_c\}$  a **chain originated at  $E_c$** . The same terminology will be used for the chains of  $\widetilde{V}$ -spaces  $\{\widetilde{V}_b, \dots, \widetilde{V}_c = \mathbb{Q}_{q,t} \widetilde{E}_c\}$ .

Concerning the relation to the spaces of generalized vectors, it is obvious that  $\widetilde{V}_c \subset \mathcal{V}(-c_\sharp)^\infty$ . The main advantage of  $\widetilde{V}_c$  versus  $\mathcal{V}(-c_\sharp)^\infty$  is that the former are defined “locally”, following a decomposition of  $\pi_c$ . The spaces of generalized eigenvectors are defined “globally”. Also, generally,  $\widetilde{V}_c$  are smaller than  $\mathcal{V}(-c_\sharp)^\infty$ . For instance, the spaces  $\mathcal{V}(-c_\sharp)^\infty$  are infinite dimensional when  $q$  is a root of unity; the spaces  $\widetilde{V}_c$  are always finite dimensional. Here a natural challenge is to make the spaces  $\{V_c\}$  generating the whole  $\mathcal{V}$  (and containing  $E_c$  in the semisimple case) as small as possible.

**9.3. The spaces  $\mathcal{V}_c$ .** A demerit of  $\widetilde{V}_c$  is that they are not limits of any natural spaces in semisimple deformations of  $\mathcal{V}$  and may depend on the reduced decompositions. We will introduce a somewhat greater system of finite dimensional spaces  $\mathcal{V}_c$  that are such limits and depend only on the corresponding  $c$ . Their pullbacks  $\mathcal{V}_c^\dagger$  will be defined in terms of the right Bruhat ordering from Section 4 and Theorem 4.2. The sequence of necessary definitions is as follows.

*First*, let us introduce a *root subsystem*  $\widetilde{R}^0 \subset \widetilde{R}$  using the relation (9.3) for  $c = 0$ :

$$(9.7) \quad \widetilde{R}^0 \stackrel{\text{def}}{=} \{\widetilde{\alpha} = [\alpha, \nu_\alpha j] \in \widetilde{R} \mid q^{\nu_\alpha j - (\alpha, \rho_k)} = q_\alpha^{j - (\alpha^\vee, \rho_k)} = 1\},$$

explicitly,  $q_\alpha^j \prod_{\nu \in \nu_R} t_\nu^{-(\alpha^\vee, \rho_\nu)} = 1 \text{ for } \rho_\nu = \frac{1}{2} \sum_{\alpha > 0, \nu_\alpha = \nu} \alpha.$

Recall that the latter product contains only integral powers of  $t_{\text{sh}}t$  and  $t_{\text{lg}}$  since  $(\rho_k, \alpha_i^\vee) = k_i = k_{\alpha_i}$  for  $i > 0$ .

Obviously  $\tilde{R}^0$  is a *root subsystem* of  $\tilde{R}$ , it is closed with respect to the addition and subtraction (if the result is a root). Note that, generally, it is not true that  $\tilde{R}^0$  is an intersection of the  $\mathbb{Q}$ -span  $\mathbb{Q}\langle\tilde{R}^0\rangle$  and  $\tilde{R}$ . Indeed, let  $0 < q < 1$  and

$$(9.8) \quad t_\nu = \zeta_\nu q_\alpha^{k_\alpha}, \quad k_\nu \in (1/N_\nu)\mathbb{Z}, \quad \zeta_\nu^{N_\nu} = 1 \quad \text{for } N_\nu \in \mathbb{N}.$$

We may assume that  $\zeta_\nu$  is a primitive  $N'_\nu$ -th root of unity for  $N'_\nu \mid N_\nu$ . Then (9.7) is equivalent to

$$(9.9) \quad j - (\alpha^\vee, \rho_k) = 0 \quad \text{and} \quad N'_\nu \mid (\alpha^\vee, \rho) \quad \text{for } \nu = \nu_\alpha.$$

The divisibility conditions  $N'_\nu \mid (\alpha^\vee, \rho)$  may not be compatible with taking fractional linear combinations of  $\tilde{\alpha} = [\alpha, \nu_\alpha j] \in \tilde{R}^0$ . Concerning *root subsystems*, see Section 4. In the absence of the roots of unity  $\zeta_\nu$ , when  $N'_\nu = 1$ , the first equation in (9.9) has a solution if and only if  $(\alpha^\vee, \rho_k)$  is in  $\mathbb{Z}_+$  (in  $\mathbb{N}$  as  $\alpha < 0$ ); generally, this involves divisibility conditions too and may be incompatible with fractional linear combinations.

*Second*, given  $c \in B$ , we take  $\pi_b \in \mathcal{B}^0(\pi_c)$ , i.e.,  $\pi_b$  is obtained from a reduced decomposition  $\pi_c = \pi_r s_{i_l} \cdots s_{i_1}$  by striking out some of the simple reflections  $s_{i_p}$  such that  $\tilde{\alpha}^p = s_{i_1} \cdots s_{i_{p-1}}(\alpha_p)$  belong to  $\tilde{R}^0$ . Then we pick certain *reduced* decompositions of the elements  $\pi_b$  and construct the corresponding polynomials  $\tilde{E}_b$ . Recall that  $\tilde{E}_b = \tilde{\Psi}_{\pi_b}(1)$ .

The **little generalized eigenspace**  $\mathcal{V}_c$  can be defined now as the linear span of  $\tilde{E}_b$  introduced above for the elements  $b \in B$ .

*Third*,  $\mathcal{V}_c$  in the semisimple case is constructed as follows. For an arbitrary one-parametric deformation  $q^l, \{t_\nu^l\}$  of  $q, t$  that makes the polynomial representation semisimple, the space  $\mathcal{V}_c^l$  is defined as the direct sum

$$\oplus_{\pi_b \in \mathcal{B}^0(\pi_c)} \mathbb{Q}_{q^l, t^l} E_b^l$$

for the Macdonald polynomials  $E_b^l$  defined for the (generic) parameters  $q^l, t^l$ .

Due to the definition of the  $E$ -polynomials and thanks to Proposition 1.6,

$$\mathcal{V}_c^l \subset (\oplus_{a \succ c} \mathbb{Q}_{q^l, t^l} X_a) \oplus \mathbb{Q}_{q^l, t^l} X_c,$$

and the projection onto the leading monomial  $X_c$  is nonzero here.

Theorem 4.2 guarantees that  $\mathcal{V}_c^l$  does not depend on the choice of the reduced decomposition of  $\pi_c$ .

**Main Theorem 9.1.** (i) The flat limit  $\{q^l \rightarrow q, t^l \rightarrow t\}$  of the space  $\mathcal{V}_c^l$  does not depend on the choice of the one-parametric deformation and coincides with  $\mathcal{V}_c$ ; this limit is defined in a sense of vector bundles over a disc at  $q, t$ . In particular,  $\mathcal{V}_c$  does not depend on the choice of the reduced decompositions of the elements  $\pi_b$  needed in the definition of  $\tilde{E}_b$ .

(ii) Given a reduced decomposition of  $\pi_c$ , the polynomials  $\tilde{E}_c$  belong to  $\oplus_{a \succeq c} \mathbb{Q}_{q,t} X_a$  and have a nonzero leading component; from now on they will be normalized by the relation

$$(9.10) \quad \tilde{E}_c - X_c \in \oplus_{a \succ c} \mathbb{Q}_{q,t} X_a.$$

The polynomials  $\{\tilde{E}_b (\pi_b \in \mathcal{B}^0(\pi_c))\}$  are linearly independent and form a basis of  $\mathcal{V}_c$  for any choices of the reduced decompositions of  $\pi_b$ .

(iii) The space  $\mathcal{V}_b$  defined for reduced  $\pi_b = s_i \pi_c$ , i.e., when  $l(\pi_b) = l(\pi_c) + 1$ , satisfies the property  $\mathcal{V}_c + \tau_+(T_i)(\mathcal{V}_c) \subset \mathcal{V}_b$  if (9.3) holds (case (b) above);  $\tilde{V}_b = \tilde{V}_c + \tau_+(T_i)(\tilde{V}_c)$  for such  $s_i$ . In particular, the space  $\tilde{V}_c$  belongs to  $\mathcal{V}_c$  for any  $c \in B$ ; also  $\dim \mathcal{V}_b > \dim \mathcal{V}_c$  and  $\dim \tilde{V}_b > \dim \tilde{V}_c$ .

(iv) The dimension of  $\mathcal{V}_b$  for a reduced  $\pi_b = s_i \pi_c$  decreases if and only if  $\Psi_i^c = \tau_+(T_i) - t_i^{1/2}$ , i.e., when

$$(9.11) \quad q_\alpha^{(\tilde{\alpha}^\vee, c_- + d) - (\alpha^\vee, \rho_k) + k_\alpha} = 1,$$

and also there exists  $\pi_a \in \mathcal{B}_o^0(\pi_c)$  such that  $s_i \pi_a \notin \pi_B$  for a certain  $a \in B$ , equivalently,  $(\alpha_i, a + d) = 0$ .

(v) For invertible  $\Psi_i^c$ ,  $\dim \mathcal{V}_b = \dim \mathcal{V}_c$  and  $\dim \tilde{V}_b = \dim \tilde{V}_c$ . When  $t_i \neq -1$  and  $\Psi_i^c = \tau_+(T_i) + t_i^{-1/2}$ , i.e. when

$$(9.12) \quad q_\alpha^{(\tilde{\alpha}^\vee, c_- + d) - (\alpha^\vee, \rho_k) - k_\alpha} = 1,$$

$\dim \mathcal{V}_b > \dim \mathcal{V}_c$  if and only if there exists  $\hat{w}' \in \mathcal{B}_o^0(\pi_c)$  such that  $\hat{w}' = s_i \pi_a$ ,  $l(\hat{w}') = l(\pi_a) + 1$  for a certain  $a \in B$  and  $\hat{w}' \notin \pi_B$ .

*Proof.* Let  $\pi_c = \pi_r s_{i_l} \cdots s_{i_1}$  be a reduced decomposition. For any  $q', t'$ , one can introduce the polynomials

$$\tilde{E}_c^{q', t'} = P_r \tilde{\Psi}_{i_l} \cdots \tilde{\Psi}_{i_p} \cdots \tilde{\Psi}_{i_1}(1),$$

where  $\tilde{\Psi}_{i_p}$  is either  $\Psi_{i_p}$  or  $\tau_+(T_{i_p})$  according to (9.5), namely,

$$\text{where either } \tilde{\alpha}^p = s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p}) \notin \tilde{R}^0 \text{ or } \tilde{\alpha}^p \in \tilde{R}^0.$$

Using the definition of  $\Psi_i$ ,

$$\Psi_{i_p} - (t_{i_p}^{1/2} - t_{i_p}^{-1/2})(Y_{\alpha_{i_p}}^{-1} - 1) = \tau_+(T_{i_p}).$$

These formulas express  $\tau_+(T_{i_p})$  in terms of the  $Y$ -intertwiners  $\Psi$ ; they hold for  $\tilde{E}_c^{q', t'}$  or for any  $\tilde{E}_b^{q', t'}$ , where  $\pi_b \in \mathcal{B}^0(\pi_c)$ . Since the Macdonald polynomials are  $Y$ -eigenvectors, we readily obtain that the polynomials  $\tilde{E}_b^{q', t'}$  belong to  $\mathcal{V}_c^l$  for such  $b$ . The limits of these polynomials are exactly  $\tilde{E}_b$  by construction.

We see that

$$(9.13) \quad \tilde{E}_c \in \oplus_{a \succeq c} \mathbb{Q}_{q,t} X_a,$$

and that the projection of  $\tilde{E}_c$  onto  $\mathbb{Q}_{q,t} X_c$  is nonzero. The next lemma makes this fact explicit and also shows that the latter claim formally follows from (9.13).

**Lemma 9.2.** *For  $b, c, \hat{w}$  satisfying  $\pi_b = \hat{w}\pi_c$  subject to  $l(\pi_b) = l(\hat{w}) + l(\pi_c)$ , i.e., if  $\pi_b = \hat{w}\pi_c$  is reduced, the monomial  $X_b$  appears in  $\tilde{\Psi}_{\hat{w}}(\tilde{E}_c)$  with a nonzero coefficient.*

*Proof.* It suffices to consider  $\hat{w} = s_i$ . Let  $(\alpha_i^\vee, c+d) > 0$ . It is straight to check (see below) that, given  $i > 0$ , the non-affine simple intertwiners  $\Psi_i$  transfers the space from (9.13) to that for  $s_i((c)) = s_i(c)$ , because so does  $T_i$ . However this argument is not applicable to  $\Psi_0$ ; this is the main part of the claim.

*First*, we assume that the intertwiner  $\Psi_i$  is not infinity, i.e.,  $\Psi_i$  is non-singular. There are two subcases,  $i > 0$  and  $i = 0$ . Let us begin with  $i > 0$ . Then  $\tilde{\Psi}_i(X_c)$  satisfies (12.35) for  $b$  and

$$\Psi_i(\tilde{E}_c) - t_i^{-1/2} X_b \in \oplus_{a \succ b} \mathbb{Q}_{q,t} X_a.$$

When  $i = 0$ , we follow the end of the proof of Theorem 5.1 from [C7] (see also Theorem 6.4 above) and come to the same relation with the coefficient  $t_i^{-1/2} q^{(c,c)/2 - (b,b)/2}$  instead of  $t_i^{-1/2}$ . Here it is sufficient to know that  $\Psi_i(\tilde{E}_c)$  is a linear combination of  $X_b$  modulo lower terms; then we only need to calculate the coefficient of  $X_b$  in  $(\tau_+(T_0))(X_c)$ .

*Second*, let  $\Psi_i$  be infinite (singular). The consideration is analogous. For  $i > 0$ , the element  $T_i$  maps

$$\oplus_{a \succeq c} \mathbb{Q}_{q,t} X_a \mapsto \oplus_{a \succeq b} \mathbb{Q}_{q,t} X_a.$$



For  $i = 0$ , we use again that the unwanted terms  $X_a$  will not appear when  $\widetilde{\Psi}_i$  is applied; then the coefficient of  $X_c$  in  $\tau_+(T_0)(X_b)$  is calculated.

□

The consideration of the leading terms readily gives that the polynomials  $\widetilde{E}_b^{q^i, t^i}$  for  $\pi_b \in \mathcal{B}^0(\pi_c)$  form a basis of  $\mathcal{V}_c^i$  and  $\{\widetilde{E}_b\}$  is a basis of the limit of  $\mathcal{V}_c^i$ . Therefore the latter does not depend on the choice of the one-parametric deformation and  $\lim \mathcal{V}_c^i = \mathcal{V}_c$ .

As a by-product, we obtain that the latter space does not depend on the choice of the reduced decomposition of  $\pi_c$ . As we will see, the interpretation of  $\mathcal{V}_c$  as a limit is not necessary here; this fact follows directly from Theorem 5.2.

Thus claims (i) and (ii) are checked; (iii) results from Theorem 4.2,(e). Claims (iv,v) are straightforward; we use that (9.12) and (9.11) are invariant under the action of  $\widetilde{W}^0$  and crossing out singular reflections in  $\pi_c$ .

For instance,  $\mathcal{V}_b$  can become smaller than  $\mathcal{V}_c$  for reduced  $\pi_b = s_i \pi_c$  if and only if there exists  $\pi_a \in \mathcal{B}_o^0(\pi_c)$  such that  $s_i \pi_a \notin \pi_B$  for a certain  $a \in B$ , equivalently,  $(\alpha_i, a + d) = 0$ , equivalently,  $\pm \pi_a^{-1}(\alpha_i)$  is a simple non-affine root. The latter means that (9.11) holds. □

**9.4. Further properties.** Let us begin with some applications to the Macdonald polynomials. Given  $c \in B$  and a reduced decomposition  $\pi_c = \pi_r s_{i_l} \cdots s_{i_1}$ , we have the following embeddings:

$$(9.14) \quad \widetilde{V}_c \subset \mathcal{V}_c \subset \mathcal{V}(-c_\sharp)^\infty.$$

The dimension of  $\mathcal{V}(-c_\sharp)^\infty$  equals one if and only if the coset  $\pi_c \widehat{W}^b[-\rho_k]$  does not contain the elements in the form  $\pi_b$  for  $b \neq c$ , where

$$\widehat{W}^b[\xi] \stackrel{\text{def}}{=} \{ \widehat{w} \in \widehat{W}^b \mid q^{\widehat{w}(\xi)} = q^\xi \}.$$

In this case, the eigenvalue  $q^{-c_\sharp}$  is  $Y$ -simple and the Macdonald polynomial  $E_c$  exists. All such  $c$  are *primary*, but, generally, the set of primary  $c = c^\circ$  is broader; see (9.2).

Note that if  $\pi_c \widetilde{W}^0$  does not contain the elements  $\pi_b$  for  $b \neq c$  for  $\widetilde{W}^0 = \langle s_{\widetilde{\alpha}} \mid \widetilde{\alpha} \in \widetilde{R}^0 \rangle$  (see Proposition 4.1), then  $\dim \mathcal{V}_c = 1$ . Since  $b \succ c$  for such  $b$ , primary  $c = c^\circ$  automatically satisfy this condition. Also  $\lambda(\pi_c) \cap \widetilde{R}^0 = \emptyset \Rightarrow \dim \mathcal{V}_c = 1$ , i.e., the dimension is one if there are no singular  $s_{i_p}$  in reduced decompositions of  $\pi_c$ .

Generally,  $\dim \mathcal{V}_c = 1$  *if and only if* there are no elements in the form of  $\pi_b$  in the set  $\mathcal{B}_o^0(\pi_c)$ . Any  $\mathcal{V}_c$  contains at least one Macdonald polynomial; indeed,  $E_{c^\circ}$  for *any* primary  $c^\circ$  (there can be several for a given element  $c$ ) can be taken.

The equality  $\dim \mathcal{V}_c = 1$  guarantees that  $E_c$  exists and does not depend on the presentation of  $\mathcal{V}$  as a limit of the generic semisimple polynomial representation  $\mathcal{V}^\lambda$ . The equality  $\dim \tilde{\mathcal{V}}_c = 1$  gives that the Macdonald polynomial  $E_c$  exists, i.e., is a  $Y$ -eigenfunction, have the required structure of its monomials and a nonzero leading term, however it may depend on the choice of the limiting procedure.

If the limit of  $\tilde{E}_c^{q^i, t^i}$  exists only for a *certain choice* of the deformation parameter then this limit is an eigenvector from the  $Y$ -eigenspace  $\mathcal{V}_c \cap \mathcal{V}(-c_\#)$ . As a matter of fact, the whole space  $\mathcal{V}_c$  with the filtration corresponding to  $>_0$  is such a limit if we switch here to more general understanding of the limiting procedure (involving induced  $\mathcal{H}^b$ -modules).

Thus:

$$(9.15) \quad \dim \tilde{\mathcal{V}}_c = 1 \Leftrightarrow \dim \mathcal{V}_c = 1 \Leftrightarrow c = c^\circ \Leftrightarrow \dim \mathcal{V}(-c_\#)^\infty = 1, \\ \dim \mathcal{V}_c = 1 \Leftrightarrow \text{existence and } \underline{\text{total}} \text{ uniqueness of } E_c.$$

Let us check that  $\dim \mathcal{V}(-c_\#)^\infty = 1$  for sufficiently big  $c \in B$  if  $q$  is not a root of unity and, therefore,  $\widehat{W}^b[-\rho_k]$  is finite. Recall that

$$\pi_c = cu_c^{-1} = u_c^{-1}c_- \quad \text{for } c_- = u_c(c) \in B_-, u \in W, \\ c_\# = c - u_c^{-1}(\rho_k) = \pi_c((-\rho_k)), \quad -0_\# = \rho_k,$$

where the affine action  $((\cdot))$  from (1.29) is used.

The following condition is sufficient for  $\dim \mathcal{V}(-c_\#)^\infty = 1$ :

$$(9.16) \quad (c_- + [\widehat{w}]_b, \alpha_i) < 0 \text{ for all } i > 0 \text{ and all } \widehat{w} \in \widehat{W}^b[\rho_k],$$

where we use the decomposition  $\widehat{w} \stackrel{\text{def}}{=} [\widehat{w}]_b[\widehat{w}]_u$  with  $[\widehat{w}]_b \in B$ ,  $[\widehat{w}]_u \in W$ . This condition means that  $c_- + [\widehat{w}]_b$  is *anti-dominant* for all  $\widehat{w} \in \widehat{W}^b[\rho_k]$ , which always holds for *sufficiently large*  $b$  provided that  $\widehat{W}^b[\rho_k]$  is finite, equivalently,  $q$  is not a root of unity. Let us check it.

First,  $[\widehat{w}]_u \neq \text{id}$  for  $\widehat{W}^b[\rho_k] \ni \widehat{w} \neq \text{id}$ ; otherwise  $[\widehat{w}]_b + \rho_k = \rho_k$  and  $[\widehat{w}]_b = 0$ . Second,

$$\pi_c \widehat{w} = (u_c^{-1}[\widehat{w}]_u)([\widehat{w}]_u^{-1}(c_- + [\widehat{w}]_b)), \quad \pi_c \widehat{w} \in \pi_B \Leftrightarrow [\widehat{w}]_u^{-1}(c_- + [\widehat{w}]_b) \in B_-.$$

The latter and relations (9.16) imply that  $[\widehat{w}]_u = \text{id}$ .

Replacing here the *complete stabilizer*  $\widehat{W}^b[-\rho_k]$  by its subgroup  $\widetilde{W}^0$ , we obtain the following *criterion* for  $\dim \mathcal{V}_c = 1$  provided that  $q$  is not a root of unity:

$$(9.17) \quad (c_- + [\widetilde{w}]_b, \alpha_i) < 0 \text{ for all } i > 0 \text{ and all } \widetilde{w} \in \widetilde{W}^0,$$

where  $\widetilde{w} = [\widetilde{w}]_b[\widetilde{w}]_u$  is defined as above.

## 10. THE STRUCTURE OF $\mathcal{V}_c$

The following proposition provides exact tools for calculating the spaces  $\mathcal{V}(-c_\#)^\infty$  and  $\mathcal{V}_c$ .

**10.1. Multiplication in  $\pi_B$ .** We continue using the decomposition  $\widehat{w} = [\widehat{w}]_b[\widehat{w}]_u$  with  $[\widehat{w}]_u \in W$  and  $[\widehat{w}]_b \in B$  and other notations from the previous section.

**Proposition 10.1.** (i) *The element  $\pi_c \widehat{w}$  for  $c \in B, \widehat{w} \in \widehat{W}^b$  can be represented in the form  $\pi_b$  if and only if the following three conditions hold for every  $\alpha \in R_+$ ,*

- (a)  $\widehat{w}(\alpha) \notin -[R_+, Z_+]$ , *equivalently,*  
 $\lambda(u) \cap \lambda(\widehat{w}) = \emptyset$ , *where*  $u = [\widehat{w}]_u$ ,
- (b) *if*  $\widehat{w}(\alpha) = [-\beta, j_\circ \nu_\beta]$  *for*  $\beta > 0, j_\circ > 0$ ,  
*then*  $[-\beta, j_\circ \nu_\beta] \notin \lambda(\pi_c)$ ,
- (c) *if*  $\widehat{w}(\alpha) = [\beta, -j_\circ \nu_\beta]$  *for*  $\beta > 0, j_\circ > 0$ ,  
*then*  $[-\beta, j_\circ \nu_\beta] \in \lambda(\pi_c)$ .

Here  $\nu_\beta = \nu_\alpha$  in (b, c). Imposing (a), the roots  $\alpha \in R_+$  satisfying (b) constitute all  $\lambda(u) \ni \alpha \notin \lambda(\widehat{w})$ , (c) describes all  $\lambda(\widehat{w}) \ni \alpha \notin \lambda(u)$ .

(ii) *Let*  $q$  *be not a root of unity. Then the elements*  $\widehat{w} \in \widehat{W}^b[-\rho_k]$  *have pairwise distinct*  $W$ -*projections*  $u = [\widehat{w}]_u$ ; *also, the elements*  $[z, j]$  *from the*  $\mathbb{Z}$ -*span*  $\widetilde{Q}^0$  *of*  $\widetilde{R}^0$  *have pairwise distinct*  $z$ -*components. For instance,*  $u = \text{id} \Leftrightarrow \widehat{w} = \text{id}$  *and, given*  $\widehat{w} \in \widehat{W}^b[-\rho_k]$ , *the component*  $j_\circ \nu_\beta$  *in (b) or (c) can be uniquely determined from the relations*

$$(b') : [\beta + \alpha, -j_\circ \nu_\beta] \in \widetilde{Q}^0, \quad (c') : [\beta - \alpha, -j_\circ \nu_\beta] \in \widetilde{Q}^0.$$

Condition (a) holds for any  $\widehat{w} \in \widehat{W}^b[-\rho_k]$  if

$$(10.18) \quad (\mathbb{Z}_+ k_{\text{sht}} + \mathbb{Z}_+ k_{\text{lng}}) \cap \mathbb{Z}_+ = \{0\}.$$

(iii) If  $q$  is not a root of unity and (a) holds for  $\widehat{w} = bu \in \widehat{W}^b[-\rho_k]$ , then  $\pi_c \widehat{w}$  is not in the form  $\pi_b$  if there exists at least one  $\alpha \in R_+$  such that  $\beta = -u(\alpha) > 0$  and also

$$(10.19) \quad [-\beta, j\nu_\beta] \in \lambda(\pi_c) \text{ for } j > 0 \text{ satisfying } [\beta + \alpha, -j\nu_\beta] \in \widetilde{Q}^0.$$

Let  $\widehat{W}^b[-\rho_k]_a$  be the set of elements  $\widehat{w}$  from  $\widehat{W}^b[-\rho_k]$  under condition (a) and  $[\widehat{W}^b[-\rho_k]_a]_u$  the set of their  $W$ -projections. If  $\lambda(\pi_c)$  contains at least one  $[-\beta, j\nu_\beta]$  for every  $\beta = -u(\alpha)$  satisfying (10.19), where  $u \in [\widehat{W}^b[-\rho_k]_a]_u$ , then  $\dim \mathcal{V}(-c_\#)^\infty = 1$ .

(iv) Given  $\widehat{w} = bu \in \widehat{W}^b[-\rho_k]_a$ , let us assume that (iii) does not hold, i.e., no  $\alpha$  exist satisfying (10.19) for  $\widehat{w}$  and  $u = [\widehat{w}]_u$ . Then  $\pi_c \widehat{w}$  is not from  $\pi_B$  if and only if there exists at least one  $\alpha \in R_+$  such that  $\beta = u(\alpha) > 0$  and

$$(10.20) \quad [-\beta, j\nu_\beta] \notin \lambda(\pi_c) \text{ for } j > 0 \text{ satisfying } [\beta - \alpha, -j\nu_\beta] \in \widetilde{Q}^0.$$

When  $\pi_c = \text{id}$ ,  $\dim \mathcal{V}(-0_\#)^\infty = 1$  if and only if  $\alpha$  satisfying (10.20) exist for the  $W$ -projection  $u = [\widehat{w}]_u$  of every element  $\widehat{w} \in \widehat{W}^b[-\rho_k]_a$ .

*Proof.* If  $\widehat{w}(\alpha) = [-\beta, -j\nu_b]$  for  $\beta > 0, j \geq 0$ , then  $\alpha \in \lambda(\widehat{w})$  but  $-\alpha \notin \widehat{w}^{-1}(\lambda(\pi_c))$  because  $-\alpha = \widehat{w}^{-1}([\beta, j_\circ \nu_\beta])$ . Thus  $\alpha$  will appear in  $\lambda(\pi_c \widehat{w})$  and the latter set cannot be in the form  $\lambda(\pi_b)$  for any  $b$ . Recall that  $\lambda(\pi_c \widehat{w})$  is obtained from  $\widehat{w}^{-1}(\lambda(\pi_c) \cup \lambda(\pi_c))$  by removing all pairs  $\{\tilde{\alpha}, -\tilde{\alpha}\}$ .

Condition (b) describes  $\alpha \in R_+$  that may appear in  $\lambda(\pi_c \widehat{w})$  because of  $\widehat{w}^{-1}(\pi_c)$ ; here  $\widehat{w}^{-1}([- \beta, j_\circ \nu_\beta]) = \alpha$  and  $[- \beta, j_\circ \nu_\beta]$  must not be from  $\lambda(\pi_c)$ .

Condition (c) gives  $\alpha$  from  $\lambda(\widehat{w})$ ; here  $\widehat{w}^{-1}([- \beta, j_\circ \nu_\beta]) = -\alpha$  and  $\alpha$  will *not* appear in  $\lambda(\pi_c \widehat{w})$  only if  $[- \beta, j_\circ \nu_\beta] \in \lambda(\pi_b)$ .

Note that if  $j > j_\circ$  under (c), then  $\widehat{w}^{-1}([- \beta, j \nu_\beta])$  becomes a positive root with the negative non-affine component  $\alpha$ ; such roots can appear in  $\pi_b$ .

The equivalence from (a) and the interpretation of (b, c) under  $\lambda(\widehat{w}) \cap \lambda(u) = \emptyset$  follow directly from  $\widehat{w}(\alpha) = bu(\alpha) = [u(\alpha), -(b, u(\alpha))]$ .

As for (ii), given  $u \in [\widehat{W}^b[-\rho_k]]_u = W \cap (\widehat{W}^b[-\rho_k] B)$ , there exists a unique  $\widehat{w} \in \widehat{W}^b[-\rho_k]$  such that  $\widehat{w} = bu$  for a certain  $b \in B$ , since  $q$  is not a root of unity. It is analogous for  $\widetilde{Q}^0$ .

Let us assume that  $(\mathbb{Z}_+ k_{\text{sht}} + \mathbb{Z}_+ k_{\text{lng}}) \cap \mathbb{Z}_+ = \{0\}$ . For instance, the conditions  $\Re(k_\nu) < 0$  for all  $\nu$  are sufficient to impose. If (a) does not hold for such  $k$ , i.e.,  $\widehat{w}(\alpha) \neq [-\beta, -j\nu_\beta]$  for  $\{\alpha > 0, \beta > 0, j \geq 0\}$ , then

$$(\alpha, -\rho_k) = ([-\beta, -j\nu_\beta], -\rho_k + d) = (\beta, \rho_k) - j\nu_\beta$$

and  $(\alpha + \beta, \rho_k) = j\nu_\beta \geq 0$ , which is impossible; see (9.7).

Claims (iii) and (iv) correspond respectively to cases (b) and (c) for  $\widehat{w} \in \widehat{W}^b[-\rho_k]$ .  $\square$

Claim (iii) from the proposition gives that the eigenspace  $\mathcal{V}(-c_\#)^\infty$  is one-dimensional for “sufficiently big”  $c$ , (iv) gives that  $\dim \mathcal{V}_c = 1$  for “sufficiently small”  $c$  (see below). We will mainly need (iii).

**Comment.** We note that (i) can be used in the theory of Schubert manifolds of the affine Grassmanian defined for the maximal parahoric subgroup in the corresponding loop group.  $\square$

Let us switch to the spaces  $\mathcal{V}_c$ , which requires changing  $\widehat{W}^b[-\rho_k]$  to  $\widetilde{W}^0$ . This reduction reduces the number of possibilities for  $\widetilde{u}$  in applications. Recall that  $\widetilde{W}^0$  is defined entirely in terms of  $\widetilde{R}^0$  from (9.7) and is, generally, simpler to control than the whole centralizer  $\widehat{W}^b[-\rho_k]$  of  $\rho_k$  in  $\widehat{W}^b$ . Also, the Bruhat ordering  $\pi_c \widetilde{u} <_0 \pi_c$  is simpler for such  $\widetilde{u}$ ; see Theorem 4.2 and Proposition 4.5,(iii).

**Corollary 10.2.** *We employ the proposition as  $q$  is not a root of unity switching to the set  $\widetilde{W}_a^0 = \widetilde{W}^0 \cap \widehat{W}^b[-\rho_k]_a$ . If for every  $u = [\widetilde{u}]_u$  for  $\widetilde{u} \in \widetilde{W}_a^0$  such that  $\pi_c \widetilde{u} <_0 \pi_c$ , the set  $\lambda(\pi_c)$  contains at least one  $[-\beta, j\nu_\beta]$  with  $\beta = -u(\alpha) > 0$  satisfying (10.19), then  $\dim \mathcal{V}_c = 1$ .*

(i) *Let us assume that  $\lambda(\pi_c)$  contains the set  $\widetilde{R}_+^1[-] = \widetilde{R}^1 \cap \widetilde{R}_+[-]$ , where*

$$(10.21) \quad \widetilde{R}_+[-] \stackrel{\text{def}}{=} \{\widetilde{\alpha} = [-\alpha, j\nu_\alpha], \alpha > 0, j > 0\} \text{ and}$$

$$(10.22) \quad \widetilde{R}^1 \stackrel{\text{def}}{=} \{\widetilde{\alpha} = [-\alpha, j\nu_\alpha] \in \widetilde{R} \mid q_\alpha^{k_\alpha + j + (\alpha^\vee, \rho_k)} = 1\}.$$

*Let  $j = 0$  in (10.22) occur for simple  $\alpha$  only, for instance, this always holds as  $t_\nu$  are not roots of unity. This condition implies that  $[-\beta, j\nu_\beta]$  from (b) belongs to  $\widetilde{R}_+^1[-]$  if and only if  $\alpha = \alpha_i$  for some  $i > 0$ . Then  $\dim \mathcal{V}_c = 1$  and, moreover,  $\lambda(\pi_c)$  contains  $\widetilde{R}^0$ .*

(ii) Continuing to impose that  $j = 0$  in (10.22) holds for simple  $\alpha$  only, let  $S$  be a subset of  $\{1, 2, \dots, n\}$ . The root lattice  $Q$  for  $R$  will be used. We assume now that

$$(10.23) \quad q^{2\rho_k^u} \notin q^Q, \text{ where } 2\rho_k^u \stackrel{\text{def}}{=} \sum_{\alpha \in \lambda(u)} k_\alpha \alpha, \text{ for all elements}$$

$$u \in W_1^S \stackrel{\text{def}}{=} \{w \in W \mid \lambda(u) \cap \{\alpha_1, \dots, \alpha_n\} = \alpha_i, i \in S\}.$$

Then  $\dim \mathcal{V}_{\mathcal{C}} = 1$  and, moreover,  $\lambda(\pi_{\mathcal{C}})$  contains  $\tilde{R}^0$  for any  $\pi_{\mathcal{C}}$  such that  $\lambda(\pi_{\mathcal{C}})$  contains all roots  $[-\beta, j\nu_\beta] \in \tilde{R}_+^1[-]$  unless  $\beta$  is in the form  $-u(\alpha_i)$  for  $i \in S$  and for  $u \in W_1^i \subset W_1^S$ .

*Proof.* The first claim is from Proposition 10.1, part (iii), where  $\tilde{u} \in \tilde{W}^0$  is taken. The statement that  $\lambda(\pi_c)$  contains  $\tilde{R}^0$  formally follows from the relation  $\dim \mathcal{V}_b = 1$  if the latter holds for all  $b$  such that  $\lambda(\pi_c) \subset \lambda(\pi_b)$ . Indeed, if  $\tilde{R}_+^0 \not\subset \lambda(\pi_c)$  then there exists  $b$  such that  $l(\pi_b) = l(\pi_b \pi_c^{-1}) + l(\pi_c)$  and  $\lambda(\pi_b) \setminus \lambda(\pi_c)$  contains singular  $\tilde{\alpha}$ . Therefore,  $\dim \mathcal{V}_b > 1$  for such  $b$  with a singular last root in  $\lambda(\pi_b)$ .

Let us check (i). The second condition concerning  $j = 0$  readily gives that  $\tilde{u}(\alpha) = [-\beta, j\nu_\beta] \in \tilde{R}_+^1[-]$  under (b) (from the proposition) holds for  $\alpha \in R_+$  and that  $\tilde{u} \in \widehat{W}[\rho_k]^b$  if and only if  $\alpha = -u^{-1}(\beta)$  is a simple root.

Let  $\tilde{u} \in \tilde{W}_a^0$ . Then there exists at least one simple  $\alpha_i \in \lambda(u)$ . Recall that (a) from Proposition 10.1 is imposed (see the definition of  $\tilde{W}_a^0$ ). Following (10.19) from (iii) (corresponding to case (b) from this proposition), we take  $-\beta = u(\alpha_i)$  for  $u = [\tilde{u}]_u$  and extend it to  $[-\beta, j\nu_\beta] \in \tilde{R}_+^1[-]$ . Since  $\lambda(\pi_c)$  contains all elements from  $\tilde{R}_+^1[-]$  by assumption, (i) is verified.

The demonstration of (ii) is a variant of the same argument. If  $\lambda(u)$  contains at least two simple roots, then we can proceed as in (i); recall that  $W_1^S$  is the set of  $w \in W$  such that  $\lambda(u)$  contains exactly one simple root  $\alpha_i$  for certain  $i \in S$ . The elements  $\pi_{\mathcal{C}} \tilde{u}$  for  $\tilde{u}$  with  $u \notin W_1^S$  cannot be represented in the form  $\pi_b$  by assumption. Concerning  $u \in W_1^S$ , the existence of its extension  $\tilde{u} = bu \in \widehat{W}^b[-\rho_k]$  is equivalent to the “ $q$ -integrality” of  $2\rho_k^u = \rho_k - u(\rho_k)$ , namely, to the relation

$$(10.24) \quad q^{\rho_k - u(\rho_k)} = q^{2\rho_k^u} \in q^Q,$$

which is not allowed in (ii) for  $i \in S$ . □

**10.2. The semisimple submodule.** Let us assume that  $q$  and  $\{t_\nu\}$  are not roots of unity and, also,  $q^a \prod_\nu t_\nu^{b_\nu} = 1$  implies  $a + \sum_\nu \nu k_\nu b_\nu = 0$ .

Then (10.22), (10.23) read respectively as

$$(10.25) \quad \tilde{R}^1 = \{ \tilde{\alpha} = [-\alpha, j\nu_\alpha] \in \tilde{R} \mid k_\alpha + j + (\alpha^\vee, \rho_k) = 0 \},$$

$$(10.26) \quad 2\rho_k^u = \sum_{\alpha \in \lambda(u)} k_\alpha \alpha \notin Q \text{ for all } u \in W_1.$$

Here  $2\rho_k^u$  belongs to  $Q$  if and only if  $\tilde{u} = bu \in \widehat{W}^b[-\rho_k]$  exists;  $\tilde{u}$  is a unique pullback of such  $u$  (if it exists).

We assume that  $q, t$  satisfy this condition in the next theorem to make its statement more transparent; generally, the “ $q$ -integrality” is sufficient to use instead of the condition for  $-\beta$  below, namely,

$$q^{-\beta + \nu_\beta \mathbb{Z}_+} \cap q^{\tilde{R}_+^1[-] + \tilde{R}_+^0[-]} = \emptyset.$$

**Theorem 10.3.** *Continuing part (i) of Corollary 10.2, we impose the inclusions  $\tilde{R}_+^1 \subset \lambda(\pi_c)$ , which result in  $\dim \mathcal{V}_c = 1$  and  $\tilde{R}_+^0 \subset \lambda(\pi_c)$ . Let  $\mathcal{V}_{ss}$  be a linear space with a basis  $E_c = \tilde{E}_c$ . We also impose (10.23) from (ii) for  $u \in W_1^i$  such that  $-\beta = u(\alpha_i) < 0$  is not a sum of the non-affine projections of the roots from  $\tilde{R}_+^1[-]$  and  $\tilde{R}_+^0[-]$ . Respectively,  $S$  will be the set of  $i$  when such  $-\beta$ , let us call them indecomposable, exist for at least one  $u \in W_1^i$ .*

*Then the space  $\mathcal{V}_{ss}$  is a  $Y$ -semisimple  $\mathcal{H}^b$ -submodule of  $\mathcal{V}$ , which is irreducible if and only if  $\tilde{R}_+^{-1} \subset \lambda(\pi_c)$  for all  $c$  corresponding to  $E_c \in \mathcal{V}_{ss}$ , where*

$$(10.27) \quad \tilde{R}^{-1} = \{ \tilde{\alpha} = [-\alpha, j\nu_\alpha] \in \tilde{R} \mid -k_\alpha + j + (\alpha^\vee, \rho_k) = 0 \}.$$

*Proof.* We have already checked the semisimplicity of  $\mathcal{V}_{ss}$ . The only possibility to get an element apart from  $\mathcal{V}_{ss}$  when applying the generators of  $\mathcal{H}^b$  to  $E_c \in \mathcal{V}_{ss}$  is when  $X_{\alpha_j}(q^{c^\sharp}) = t_j$  for some  $j \geq 0$ . Indeed, it can happen only when  $s_j \pi_c$  is *not* a reduced decomposition (then  $s_j \pi_c$  can be represented in the form  $\pi_{c'}$ ) with  $s_j$  corresponding to a non-invertible intertwiner  $\Psi_j^c$ . The latter cannot be  $\tau_+(T_j) - t_j^{1/2}$  because in this case the condition  $\tilde{R}_+^1[-] \subset \lambda(\pi_c)$  remains unchanged for  $\pi_{c'} = s_j \pi_c$  (if the latter belongs to  $\pi_B$ ). Thus it must be  $\tau_+(T_j) + t_j^{-1/2}$ , equivalently,  $X_{\alpha_j}(q^{c^\sharp}) = t_j$ .

Since  $\dim \mathcal{V}_c = 1$ , we can assume that  $s_j$  is the last simple reflection in a reduced decomposition of  $\pi_c$  and the *last root*  $[-\beta, j\nu_\beta]$  in  $\lambda(\pi_c)$  belongs to  $\tilde{R}_+^1$ . If  $\dim \mathcal{V}_{c'} = 1$  for (non-reduced)  $\pi_{c'} = s_j \pi_c$  above, then  $(\tau_+(T_j) + t_j^{-1/2})(E_c) = 0$  and  $\tau_+(T_j)(E_c) \in \mathcal{V}_{ss}$ , i.e., we stay within  $\mathcal{V}_{ss}$ .

Generally,  $\mathcal{V}_{c'}$  can be of dimension greater than 1 because one root from  $\tilde{R}_+[-]$ , namely  $[-\beta, j\nu_\beta]$ , may be missing in  $\lambda(\pi_{c'})$ . The component  $-\beta$  of this root is *indecomposable* (see above). Indeed, if the root  $[-\beta, \nu_\beta j] \in \tilde{R}_+[-]$  is the *last root* in  $\lambda(\pi_c)$ , then it is *not* a sum of two roots in this set, which contains  $\tilde{R}_+^1[-]$  by construction and  $\tilde{R}_+^0$  due to Corollary 10.2(i).

Now let us follow part (ii) of this corollary. We use that  $\dim V_{c'} = 1$  if for *any*  $u \in [\tilde{W}^0]_u$  we can find *at least one*  $\alpha \in R_+$  satisfying (10.19). The only problem with finding such  $\alpha$  may occur when  $\lambda(u)$  contains precisely one *simple* root  $\alpha_i$ . In this case,  $\dim = 1$  if  $\lambda(\pi_{c'})$  contains  $\tilde{u}(\alpha_i)$  assuming that the extension of  $u$  to  $\tilde{u} = ub \in \tilde{W}^0$  exists. However only one root,  $[-\beta, j\nu_\beta]$ , from  $\tilde{R}_+^1[-]$  can be absent in  $\lambda(\pi_{c'})$ . Therefore,  $\dim V_{c'} = 1$  unless  $[-\beta, j\nu_\beta] = \tilde{u}(\alpha_i)$ . We do not allow the latter simply imposing the condition that such  $u$  do not have extensions in  $\tilde{W}^0$ .

Due to the semisimplicity of  $\mathcal{V}_{ss}$ , its irreducibility is *equivalent* to the absence of non-invertible intertwiners in *reduced* decompositions of  $\pi_b$  after a certain  $\pi_c$  corresponding to  $E_c \in \mathcal{V}_{ss}$ .  $\square$

The condition  $\tilde{R}_+^1[-] \subset \lambda(\pi_c) \Rightarrow \tilde{R}_+^{-1}[-] \subset \lambda(\pi_c)$ , which ensures the irreducibility of  $\mathcal{V}_{ss}$ , holds almost always, for instance, in the simply-laced case. See Lemma 12.1 below.

When  $k_{\text{sht}} = k = \nu_{\text{lng}} k_{\text{lng}}$ , only the roots with the smallest possible  $(\beta, \rho^\vee)$  satisfying  $k((\beta, \rho^\vee) + 1) \in \mathbb{N}$  can be the *last roots* in  $\lambda(\pi_c)$  provided that this set contains  $\tilde{R}_+^1[-]$ . The constraint (a) from Proposition 10.1 simply means that  $k < 0$  in this case (only rational  $k < 0$  are sufficient to consider); otherwise,  $\tilde{R}^0 = \emptyset$  and the polynomial representation is semisimple irreducible. Note that

$$\tilde{R}_+^1 \stackrel{\text{def}}{=} \tilde{R}^1 \cap \tilde{R}_+ = \tilde{R}_+^1[-] \cup \{\alpha_i, i > 0\} \text{ since } k < 0.$$

We mention that there are other ways to justify that  $\mathcal{V}_{ss}$  is a  $\mathcal{H}^b$ -submodule (especially, for equal  $k$ ); however, always a reduced decomposition of the elements  $\pi_c$  can be found for  $E_c$  appearing in  $\mathcal{V}_{ss}$  such that  $\dim \mathcal{V}_{c'} = 1$  occurs for  $\pi_{c'}$  “before”  $\pi_c$  (in this decomposition).



**The case of  $A_n$ .** Let connect our  $\mathcal{V}_{ss}$  in the case of  $A_n$  with the construction from [Ka], which extends that from [FJMM]. In the latter paper, a symmetric variant of  $\mathcal{V}_{ss}$  was defined (for  $GL_{n+1}$ ). Namely, for an arbitrary negative rational  $k$  (with the denominator no greater than  $n+1$ ), a set of weights was found such that the corresponding symmetric Macdonald polynomials exist and linearly generate the space closed with respect to the multiplication. The authors note that the symmetric Macdonald polynomials actually exist for a bigger set of weights. As a matter of fact, this remark is closely connected with our approach; we prove that  $\mathcal{V}_{ss}$  is a DAHA-submodule using that  $\mathcal{V}_{c'}$  become one-dimensional *before*  $c'$  reach the set  $\{c\}$  corresponding to  $\{E_c\}$  linearly generated  $\mathcal{V}_{ss}$ .

Paper [Ka] contains a statement equivalent to our one in the  $A_n$ -case. We note that the technique of intertwiners and the analysis of  $\mathcal{V}_{ss}$  is significantly simpler for  $A_n$  than for other root systems. In [Ka],  $\mathcal{V}_{ss}$  was a part of the conjectural decomposition of  $\mathcal{V}$  in terms of irreducible modules (the Kasatani conjecture) justified in [En] for  $r \neq 2$  using the localization functor for the degenerate DAHA of type  $A$ .

Let  $t = q^k$ ,  $k = -s/r$  and  $(r, s) = 1, n+1 \geq r > 1$  provided that  $q^{at^b} = 1$  for  $a, b \in \mathbb{Q}$  implies  $a + kb = 0$ .

Using the notation from [B],  $\alpha_i = \epsilon_i - \epsilon_{i+1}$ ,

$$2\rho = n\epsilon_1 + (n-2)\epsilon_2 + \dots + (n-2(j-1))\epsilon_j + \dots + (-n)\epsilon_{n+1}.$$

Given  $1 \leq i \leq n$ , let us determine all permutations  $u \in W_1^i$  that can be lifted to elements  $ub$  from  $\hat{w}[-\rho_k]$ . Such permutations can be described as partitions of the segment  $[1, \dots, n]$  in terms of  $2m > 2$  consecutive (connected) *segments*  $L_p = [p', p'']$  with  $|L_p| = p'' - p' + 1$  elements:

$$(10.28) \quad L_1, L_2, L_3, \dots, i, \dots, L_{2m-2}, L_{2m-1}, L_{2m} \ni n, \\ \text{where } |L_p| > 0, |L_p| = 0 \bmod r \text{ for } p = 2, 3, \dots, 2m-1, \\ \text{and } |L_1| + |L_3| + |L_5| + \dots + |L_{2m-1}| = i, \quad L_p = [p', p'']$$

Note that only  $L_1, L_{2m}$  are allowed to be empty. The corresponding permutation  $u$  is

$$\{L_1, L_3, L_5, \dots, L_{2m-3}, L_{2m-1}, L_2, L_4, \dots, L_{2m-2}, L_{2m}\}.$$

This  $u$  sends  $\alpha_i = \epsilon_i - \epsilon_{i+1}$  to  $-\beta = \epsilon_x - \epsilon_y$  for  $x = (2m-1)'', y = 2'$  and  $(\beta, \rho) = ((n-2(y-1)) - (n-2(x-1)))/2 = x-y$ ; here  $x > y$ . The relation  $(\beta + \alpha_i, \rho_k) \in \mathbb{Z}$  (see  $(b, b')$  from Proposition 10.1), which guaranties that  $-\beta$  is a projection of the element from  $\tilde{R}_+^1[-]$ , gives that

the cardinalities of the segments between  $L_2$  and  $L_{2m-1}$  are divisible by  $r$  and  $x - y + 1$  is divisible by  $r$ .

Since  $m > 1$ ,  $x - y + 1$  must be at least  $2r$  that means that  $-\beta$  is always decomposable in the sense of Theorem 10.3. Thus  $\mathcal{V}_{ss}$  is always an  $\mathcal{H}$ -submodule in the  $A$ -case when  $r > 1$ .

*A counterexample for  $A_1$ .* The simplest example when  $\mathcal{V}_{ss}$  is *not* a  $\mathcal{H}$ -submodule of  $\mathcal{V}$  is for  $\tilde{R} = \tilde{A}_1$  and  $k = -m \in -\mathbb{N}$ ; the relation  $\dim \mathcal{V}_{c'} > 1$  does not hold in this case. Here  $u = s_1$  can be lifted to  $\tilde{u} = [-\alpha_1, m] \in \tilde{W}^0 \simeq S_2$  and  $[-\beta, \nu_\beta j] = \tilde{u}(\alpha_1) = [-\alpha_1, 2m]$ . The latter root is the only element of  $\tilde{R}_+^1[-]$ . The condition  $k + j + (\beta, \rho_k) = 0$  from the definition of  $\tilde{R}_+^1[-]$  in (10.25) reads as:  $-m + 2m - m(\alpha_i, \rho) = 0$ .

The space  $\mathcal{V}_c = \tilde{V}_c$  becomes one-dimensional for  $\pi_c = (s_1 s_0)^m$ ; here  $\lambda(\pi_c) = \{[-\alpha_1, 2m], \dots, [-\alpha_1, 1]\}$  and the last intertwiner in the chain is  $\tau_+(T_1) - t^{1/2}$ . The spaces  $\mathcal{V}_{c'}$  are two-dimensional for previous  $\pi_{c'} = \dots s_0 s_1 s_0$ , i.e., when the number  $l$  of simple reflection in this product satisfies  $2m - 1 \geq l \geq m$ ; here  $\lambda(\pi_{c'}) = \{[-\alpha_1, l], \dots, [-\alpha_1, 1]\}$ .

The polynomial representation is irreducible but not semisimple for such  $k$ . Note that there are no *positive* affine roots  $\tilde{\alpha}$  in this case such that  $-k + (\tilde{\alpha}, -\rho_k + d) = 0$ ; the roots  $-\alpha_1$  and  $[\alpha_1, -2m]$  satisfying this condition are negative. This means that  $\tilde{R}_+^{-1}[-]$  is empty and  $\tau_+(T_i) + t^{-1/2}$  will not appear in the chains of intertwiners in  $\mathcal{V}$ .  $\square$

**10.3. Eigenspaces for  $\pi_c \tilde{w}$ .** The following is another application of Proposition 10.1, which is important for *exact* calculating  $\dim \mathcal{V}_c$  and  $\dim \mathcal{V}[-c_\sharp]^\infty$ . For instance, it can be applied to reinstate Theorem 10.3.

We continue using the notation  $\tilde{w} = [\tilde{w}]_b [\tilde{w}]_u$ . Let  $\lambda_R^1(\tilde{w})$  be the subset of *simple non-affine* roots in  $\lambda(\tilde{w})$ , that is  $\lambda(\tilde{w}) \cap R^1$ , where we set  $R^1 \stackrel{\text{def}}{=} \{\alpha_i, 1 \leq i \leq n\}$ . When  $q$  is not a root of unity,

$$(10.29) \quad \lambda_R^1(\tilde{w}) = \{\alpha_i, i > 0 \mid [\tilde{w}]_u(\alpha_i) > 0, [\tilde{w}]_u(\alpha_i) \neq \alpha_{i'}, i' > 0\}.$$

The set  $\tilde{R}^{\pm 1}$  can be either as in Theorem 10.3 under

$$q^a \prod_{\nu} t_{\nu}^{b_{\nu}} = 1 \Rightarrow a + \sum_{\nu} \nu k_{\nu} b_{\nu} = 0,$$

or as in Proposition 10.1 without this assumption.

We constantly use that  $\tilde{W}^0$  and  $\widehat{W}^b[-\rho_k]$  preserve  $\tilde{R}^{\pm 1}$ . For instance,  $\tilde{w}(R^1 \setminus \lambda_R^1(\tilde{w})) \subset \tilde{R}^1$  and  $-\tilde{w}(\lambda_R^1(\tilde{w})) \subset \tilde{R}^{-1}$  in the Main Theorem, (i) below.

**Main Theorem 10.4.** (i) Given  $\tilde{w} \in \tilde{W}^0$ , we impose condition (a) from (i) in Proposition 10.1 for every  $\alpha \in R_+$ ; it is necessary to ensure  $\pi_c \tilde{w} \notin \pi_B$  (recall that (a) results from (10.18)). Setting  $\lambda_R^1 = \lambda_R^1(\tilde{w})$ ,

$$(10.30) \quad \pi_c \tilde{w} \in \pi_B \Leftrightarrow \{-\tilde{w}(\lambda_R^1) \subset \lambda(\pi_c), \tilde{w}(R^1 \setminus \lambda_R^1) \cap \lambda(\pi_c) = \emptyset\}.$$

Let us take  $\tilde{w}$  such that  $\pi_c \tilde{w} \in \mathcal{B}_o^0(\pi_c)$ , i.e.,  $\pi_c \tilde{w}$  must be obtained from a reduced decomposition of  $\pi_c$  by deleting some singular reflections. Then

$$(10.31) \quad \mathcal{V}_c = \sum_b \mathbb{Q}_{q,t} \tilde{E}_b \text{ for } \pi_b = \pi_c \tilde{w} \in \mathcal{B}_o^0(\pi_c).$$

(ii) Let us consider a reflection  $\tilde{w} = s_{\tilde{\alpha}}$  in (i) for  $\tilde{\alpha} = [-\alpha, \nu_{\alpha} j] \in \tilde{R}_+^0$ , where  $\alpha > 0, j > 0$ . Then (a) holds,  $[\tilde{w}]_u = s_{\alpha}$  and

$$\lambda_R^1(\tilde{w}) = \{\alpha_{i'} \mid (\alpha_{i'}, \alpha) < 0, i' > 0\}.$$

Let  $\pi_c \tilde{w}$  be the result of deleting one singular simple reflection  $s_p$  in a reduced decomposition of  $\pi_c$ . Then  $\pi_c \tilde{w} = \pi_b$ , equivalently  $\tilde{E}_b \in \mathcal{V}_c$ , if and only if

$$(10.32) \quad s_{\tilde{\alpha}}(\alpha_{i'}) = \alpha_{i'} + 2 \frac{(\alpha, \alpha_{i'})}{(\alpha, \alpha)} \tilde{\alpha} \notin \lambda(\pi_c) \text{ for all } (\alpha_{i'}, \alpha) > 0;$$

here the condition  $\{-\tilde{w}(\lambda_R^1) \subset \lambda(\pi_c)\}$  holds automatically. If  $s_{\tilde{\alpha}}(\alpha_{i'}) \in \lambda(\pi_c)$  for  $i' > 0$  satisfying (10.32), then  $\pi_c s_{\tilde{\alpha}} \notin \pi_B$  and this root can appear in the sequence  $\lambda(\pi_c)$  only after (the root corresponding to)  $s_p$ ; For instance, (10.32) holds if  $s_p$  is the last in the decomposition of  $\hat{w}$ .

(iii) If  $(\alpha_{i'}, \alpha) < 0$  for  $i' > 0$ , then  $-s_{\tilde{\alpha}}(\alpha_{i'}) \in \tilde{R}_+^{-1}[-]$  appears in  $\lambda(\pi_c)$  before  $s_p$ . For arbitrary  $\tilde{\alpha}$  (not associated with  $s_p$ ), the conditions  $\lambda(\pi_c) \ni -s_{\tilde{\alpha}}(\alpha_{i'}) \in \tilde{R}_+^{-1}[-]$  and (10.32) are necessary and sufficient to ensure that  $\pi_c s_{\tilde{\alpha}} \in \pi_B$ . More generally, provided (a) for  $\hat{w}$  as in (i),

$$(10.33) \quad \mathcal{V}(-c_{\sharp})^{\infty} = \sum_b \mathbb{Q}_{q,t} \tilde{E}_b \text{ for } \pi_b = \pi_c \hat{w}, \hat{w} \in \widehat{W}^b[-\rho_k], \text{ where}$$

$$\pi_c \hat{w} \in \pi_B \Leftrightarrow \{-\hat{w}(\lambda_R^1(\hat{w})) \subset \lambda(\pi_c), \hat{w}(R^1 \setminus \lambda_R^1(\hat{w})) \cap \lambda(\pi_c) = \emptyset\}.$$

If  $\hat{w}$  does not satisfy (a) from Proposition 10.1 or one of the latter conditions in  $\{ , \}$  does not hold for every  $\hat{w} \in \widehat{W}^b[-\rho_k]$  then  $\dim \mathcal{V}(-c_{\sharp})^{\infty} = 1$ ; vice versa, the condition  $\dim = 1$  implies that all three must hold.

*Proof.* It is a version of Proposition 10.1 where we use that  $\hat{v} \notin \pi_B$  implies that  $\lambda(\hat{v})$  contains at least one *simple* non-affine root. i.e., a

root in the form  $\alpha_{i'}$  for  $i' > 0$ . We use (1.20) to check that  $\{-\tilde{w}(\lambda_R^1) \subset \lambda(\pi_c)\}$  holds automatically in this case.

The justification of claims about the position of  $s_{\tilde{\alpha}}(\alpha_{i'})$  from (ii) and  $-s_{\tilde{\alpha}}(\alpha_{i'})$  from (iii) respectively after and before  $s_p$  is as follows. Let  $s_{\tilde{\alpha}}(\alpha_{i'}) = \alpha_j + 2\frac{(\alpha, \alpha_{i'})}{(\alpha, \alpha)}\tilde{\alpha}$  assuming that  $i' > 0$ ,  $(\alpha, \alpha_{i'}) > 0$ . Then  $s_{\tilde{\alpha}}(\alpha_{i'})$  is a sum of  $\alpha_{i'}$  and  $\tilde{\alpha} \in \lambda(\pi_c)$  with positive coefficients. Since  $\alpha_{i'} \notin \lambda(\pi_c)$ ,  $\tilde{\alpha}$  must appear before  $s_{\tilde{\alpha}}(\alpha_{i'})$  for any reduced decomposition of  $\pi_c$ . See Theorem 2.1. The claim from (iii) is analogous.  $\square$

**Spaces  $\tilde{\mathcal{V}}_c$ .** Given an arbitrary, *non-necessarily reduced*, decomposition of  $\hat{w}$ , the set  $\tilde{\mathcal{B}}^0(\hat{w}) \ni \hat{w}'$  was introduced in Theorem 5.2 and the corresponding **standard** decompositions of  $\hat{w}'$  obtained by deleting simple singular reflections. This set has a natural partial ordering. Recall that

$$(10.34) \quad \tilde{E}_c^\dagger \stackrel{\text{def}}{=} \tilde{\Psi}_{\hat{w}}(1) \in \oplus_{a \succeq c'} \mathbb{Q}_{q,t} X_a \quad \text{for } c' = \hat{w}'((c)),$$

where  $\hat{w}' \in \tilde{\mathcal{B}}^0(\pi_c)$ . Without  $\dagger$ , the  $\tilde{E}$ -polynomials are

$$\tilde{E}_c = \text{Const } P_r \tilde{\Psi}_{i_l} \cdots \tilde{\Psi}_{i_1}(1) \quad \text{for } \pi_c = \pi_r s_{i_l} \cdots s_{i_1},$$

where the reduced decomposition is used and the constant is chosen to ensure the normalization (9.10), i.e., the coefficient of  $X_c$  has to be 1. If the decomposition of  $\pi_c$  is not assumed to be reduced (the  $\dagger$ -case), then  $\tilde{E}_c^\dagger$  are defined only up to proportionality and they can be zero. The spaces  $\tilde{\mathcal{V}}_c$  and  $\mathcal{V}_c$  are constructed only for the reduced decompositions of  $\pi_c$  in the following proposition and later.

Recall that  $\tilde{\mathcal{V}}_c$  is the linear span of  $\tilde{E}_b^\dagger$  for  $b = \hat{w}'((0))$  defined for all *standard decompositions* of  $\hat{w}' \in \tilde{\mathcal{B}}^0(\pi_c)$ , *possibly non-reduced*, obtained from a given reduced decomposition of  $\pi_c$ . The space  $\mathcal{V}_c$  is linearly generated by  $\hat{E}_b$  for the same set  $\{b\}$ , however they are defined only for *reduced* decompositions of  $\pi_b$ ; they are linearly independent for any particular choice of these decompositions.

Practically, the difference between  $\mathcal{V}_c$  and  $\tilde{\mathcal{V}}_c$  is as follows. The squares  $\dots(\Psi_i)^2\dots$  (they are  $Y$ -rationals) in the formulas for  $\tilde{E}_c^\dagger$  and  $\tilde{\mathcal{V}}_c$  are replaced by their  $Y$ -expressions; such squares are simply deleted when constructing  $\mathcal{V}_c$ .

Note that the linear span of  $\tilde{\Psi}_{\hat{w}}\tilde{V}_c$  for *all* decompositions, possibly non-reduced, of  $\hat{w} \in \widehat{W}^b$  is an  $\mathcal{H}^b$ -submodule of  $\mathcal{V}$ .

As we know, the space  $\mathcal{V}_c$  does not depend on the choice of the reduced decomposition of  $\pi_c$ . It follows from the limiting procedure or can be readily checked using Theorem 5.2 (apply  $\tau_-\sigma$  to  $\Phi_{\hat{w}}$  there and use  $\tilde{\Psi}_i = \tau_+(T_i)$  for singular  $s_i$  in  $\tilde{\Psi}_{\hat{w}}$ ).

Individual  $\tilde{E}_c$  and the space  $\tilde{V}_c$  defined for a given reduced decomposition of  $\pi_c$  may change if the homogeneous Coxeter transformations of type (5.17) from Theorem 5.2,(a) are applied. However formula (5.19) makes it possible to control the change in this case.

**Proposition 10.5.** *The spaces  $\mathcal{V}_c$  and  $\tilde{V}_c$  are  $\mathbb{Q}_{q,t}[Y_b]$ -modules and are also invariant with respect to the action of the operators  $\tau_+(T_i)$  for  $s_i$  such that  $\pi_c^{-1}(\alpha_i) \in \tilde{R}^0$  and  $l(s_i\pi_c) < l(\pi_c)$ . If  $q, t$  are not roots of unity and the whole centralizer of  $q^{-c_\#}$  in  $\widehat{W}^b$  is generated by simple reflections  $s_i$ , then*

$$\tilde{V}_c = \mathcal{V}_c = \mathcal{V}(-c_\#)^\infty.$$

*Proof.* See Theorem 9.1,(iii) and Theorem 4.2,(e). The last claim follows from the fact that  $\mathcal{V}(-c_\#)^\infty$  is an induced module over the affine Hecke algebra defined for  $\tilde{R}^0$ , that is irreducible and  $Y$ -cyclic in this case; cf. Proposition 10.11 below.  $\square$

**10.4. The  $Y$ -action.** The following theorem makes the  $Y$ -structure of  $\tilde{V}_c$  or  $\mathcal{V}_c$  as explicit as possible (the latter is somewhat simpler to calculate than the former). Note that the formulas below do not give a complete description of the  $Y$ -action because the polynomials  $\hat{E}_c^\dagger$  that appear in process of calculations can be zero or linearly dependent in  $\tilde{V}_c \subset \mathcal{V}_c$ .

**Main Theorem 10.6.** *Given  $b \in B$ , an element  $c \in B$  and also its decomposition  $\pi_c = \pi_r s_{i_1} \cdots s_{i_1}$ , non-necessarily reduced, let  $\{i_p\}$  be the set of singular indices in this decomposition for  $p \in \{p_g, \dots, p_1\}$ . Then*

$$(10.35) \quad q^{(b, c_\#)} Y_b(\tilde{E}_c^\dagger) = \tilde{E}_c^\dagger + \sum_{c'} \prod_{p'=p'_j}^{1 \leq j \leq g'} (t_{i_{p'}}^{1/2} - t_{i_{p'}}^{-1/2}) C_{c'} \tilde{E}_{c'}^\dagger,$$

where  $c' = \widehat{w}'((0))$ ,  $\widehat{w}' \in \widetilde{\mathcal{B}}^0(\pi_c)$  and  $\widetilde{E}_{c'}^\dagger$  are defined for the decompositions, possibly non-reduced, of  $\widehat{w}'$  obtained from the initial decomposition of  $\pi_c$  by deleting singular  $s_{i_{p'}}$  for the indices  $p'$  forming a subsequence

$$\{p'_{g'}, \dots, p'_1\} \subset \{p_g, \dots, p_1\}.$$

For instance, if there is only one such  $p'$ , i.e.,  $g' = 1$ , then

$$(10.36) \quad \begin{aligned} C_{c'} &= ((\widetilde{\beta}^{p'})^\vee, b + d) \text{ for } \widetilde{\beta}^{p'} \stackrel{\text{def}}{=} (\pi_r s_{i_1} \cdots s_{i_{p'+1}})(\alpha_{i_{p'}}), \\ ((\widetilde{\beta}^{p'})^\vee, b + d) &= -((\widetilde{\alpha}^{p'})^\vee, \pi_c^{-1}((b)) + d) \text{ for } \widetilde{\alpha}^{p'} = s_{i_1} \cdots s_{i_{p'-1}}(\alpha_{i_{p'}}). \end{aligned}$$

Generally, when the number of the indices  $p'$  is  $h = g' \geq 1$ , we set  $\widehat{b} = \pi_c^{-1}((b))$  and the formula for  $C_{c'}$  reads as follows:

$$(10.37) \quad \begin{aligned} C_{c'} &= (Y_{\widehat{b}}^{-1}(D_{\widetilde{\alpha}^{p'_1}}^Y \cdots D_{\widetilde{\alpha}^{p'_h}}^Y)(Y_{\widehat{b}})) \Downarrow 1^0 = (D_{\widetilde{\alpha}^{p'_1}}^Y \cdots D_{\widetilde{\alpha}^{p'_h}}^Y)(Y_{\widehat{b}}) \Downarrow 1, \\ \text{setting } Y_{\widetilde{\alpha}} \Downarrow 1^0 &= 1 \text{ for } \widetilde{\alpha} \in \widetilde{R}^0, \ Y_a \Downarrow 1 = 1 \text{ for all } a \in B, \\ \text{where } D_{\widetilde{\alpha}}^Y &\stackrel{\text{def}}{=} (Y_{\widetilde{\alpha}} - 1)^{-1}(s_{\widetilde{\alpha}}^Y - 1) \text{ for } \widetilde{\alpha} \in \widetilde{R}, \end{aligned}$$

defined in terms of the action  $s_{\widehat{w}}^Y(Y_a) \stackrel{\text{def}}{=} Y_{\widehat{w}(a)}$  for  $\widehat{w} \in \widehat{W}$ .

*Proof.* We use the  $\tau_- \sigma$ -image of the relations (6.1):

$$(10.38) \quad Y_b \tau_+(T_i) = \tau_+(T_i) Y_{s_i(b)} + (t_i^{1/2} - t_i^{-1/2}) \frac{s_i^Y(Y_b) - Y_b}{Y_{\alpha_i}^{-1} - 1}.$$

Recall that  $\tau_+(T_i) = \tau_+ \tau_-^{-1}(T_i) = \tau_- \sigma(T_i)$  and  $\tau_- \sigma(X_b) = Y_b^{-1}$ .

Let us introduce the operators

$$(10.39) \quad \mathcal{T}_{\widetilde{\alpha}}^Y \stackrel{\text{def}}{=} t_{\alpha}^{1/2} s_{\widetilde{\alpha}}^Y - \frac{t_{\alpha}^{1/2} - t_{\alpha}^{-1/2}}{Y_{\widetilde{\alpha}}^{-1} - 1} (s_{\widetilde{\alpha}}^Y - 1),$$

satisfying the relations

$$(10.40) \quad Y_b \mathcal{T}_{\widetilde{\alpha}}^Y = \mathcal{T}_{\widetilde{\alpha}}^Y Y_{s_i(b)} + \frac{t_{\alpha}^{1/2} - t_{\alpha}^{-1/2}}{Y_{\widetilde{\alpha}}^{-1} - 1} (s_{\widetilde{\alpha}}^Y - 1)(Y_b).$$

For instance,  $\mathcal{T}_i^Y = \mathcal{T}_{\alpha_i}^Y$  satisfy (10.38) for  $\tau_+(T_i)$ .

These relations readily give (10.37). One needs to move  $Y_b$  in  $Y_b \widetilde{\Psi}_{\widehat{w}}$  through  $\widetilde{\Psi}_{\widehat{w}}$  and calculate the coefficients of  $\widetilde{\Psi}_{\widehat{w}}^\dagger Y_a$  for  $\widehat{w}' \in \widetilde{\mathcal{B}}^0(\widehat{w})$  in the resulting decomposition. Here using  $^\dagger$  in  $\widetilde{\Psi}_{\widehat{w}}^\dagger$  indicates that when some simple intertwiners  $\widetilde{\Psi}_i$  disappear from the product due to (10.40),

we leave the product as it is (without reducing the corresponding decompositions).

These coefficients *coincide* with those obtained when  $Y_b$  is moved through  $\widehat{w}^Y \mathcal{R}_{\tilde{\alpha}^{p_g}}^Y \cdots \mathcal{R}_{\tilde{\alpha}^{p_1}}^Y$  (instead of  $\tilde{\Psi}_{\widehat{w}}$ ), where  $\mathcal{R}_{\tilde{\alpha}}^Y \stackrel{\text{def}}{=} s_{\tilde{\alpha}}^Y T_{\tilde{\alpha}}^Y$  satisfy

$$(10.41) \quad Y_b \mathcal{R}_{\tilde{\alpha}}^Y = \mathcal{R}_{\tilde{\alpha}}^Y Y_b + (t_{\alpha}^{1/2} - t_{\alpha}^{-1/2}) D_{\tilde{\alpha}}^Y(Y_b).$$

The product  $Y_b \widehat{w}^Y \mathcal{R}_{\tilde{\alpha}^{p_g}}^Y \cdots \mathcal{R}_{\tilde{\alpha}^{p_1}}^Y$  will become a linear combination of the terms

$$\widehat{w}^Y \mathcal{R}_{\tilde{\alpha}^{p'_h}}^Y \cdots \mathcal{R}_{\tilde{\alpha}^{p'_1}}^Y Y_a \quad \text{for } \widehat{w}' \in \tilde{\mathcal{B}}^0(\widehat{w}),$$

where we do not perform reductions if some of these terms are linearly dependent.

Then we apply the evaluation  $\Downarrow 1^0$ ; here one can also take  $\Downarrow 1$ , sending  $Y_a \mapsto 1$  for all  $a$ , because  $\{\tilde{\alpha}^{p'}\}$  are from  $\tilde{R}^0$ .

In the case of one  $p'$ , the coefficient  $C_{c'}$  from (10.36) is  $D_{\tilde{\alpha}^{p'}}^Y(Y_{\widehat{w}^{-1}((b))})$  upon the evaluation  $\Downarrow 1$  for  $\widehat{w} = \pi_c$ ; it equals

$$\begin{aligned} & -((\tilde{\alpha}^{p'})^\vee, \widehat{w}^{-1}((b)) + d) = -(\widehat{w} s_{i_1} \cdots s_{i_{p'-1}} (\alpha_{i_{p'}})^\vee, b + d) \\ & = -(\pi_r s_{i_1} \cdots s_{i_{p'}} (\alpha_{i_{p'}})^\vee, b + d) = ((\tilde{\beta}^{p'})^\vee, b + d). \end{aligned}$$

□

Note that  $C_{c'}$  are integers. Their calculation is a combinatorial problem that can be formulated in terms of the algebra generated by the **affine Demazure operators**  $D_{\tilde{\alpha}}^Y$  defined for  $\tilde{\alpha} \in \tilde{R}^0$ . They satisfy the  $r$ -matrix relations and the quadratic ones:  $D_{\tilde{\alpha}}^Y(D_{\tilde{\alpha}}^Y - 1) = 0$ . The affine Hecke algebra for the root system  $\tilde{R}^0$  is sufficient for calculating these coefficients. The degenerate affine Hecke algebra can be used here, which is useful for analyzing the  $Y$ -cyclicity of  $\tilde{V}_c$ . Moreover, almost always a reduction to the *non-affine* Hecke algebra is sufficient.

The elements  $\tilde{E}_{c'}^\dagger$ , when considered as vectors in  $\tilde{V}_c$ , can vanish and some can become linearly dependent. The next stage of the calculation is when we express the  $\tilde{E}^\dagger$ -polynomials in terms of *reduced*  $\tilde{E}$ -polynomials. Replacing  $T_i^2$  using the quadratic relations may be necessary and moving the  $Y$ -functions  $\Psi_i^2$  to the right through  $\Psi$  and  $T$ . Eventually, some  $\tilde{E}_{c'}^\dagger \neq 0$  may vanish and some may become linearly dependent. It is convenient to perform this calculation inside  $\mathcal{V}_c$ .

The following “rationalization” of (10.35) is needed for the complete description of  $\mathcal{V}_c$  as  $Y$ -modules.

Given a rational function  $P_Y$  of  $Y_b$  such that  $P_Y(1)$  is well defined, setting  $\widehat{P}_Y = (\widehat{w}^Y)^{-1}(P_Y)$ :

$$(10.42) \quad P_Y(\widetilde{E}_c^\dagger) = \widetilde{E}_c^\dagger P_Y(1) + \sum_{c'} \widetilde{E}_{c'}^\dagger \left( \prod_{\substack{1 \leq j \leq h=g' \\ p'=p'_j}} (t_{i_{p'}}^{1/2} - t_{i_{p'}}^{-1/2}) \right) (D_{\widetilde{\alpha}^{p'_1}}^Y \cdots D_{\widetilde{\alpha}^{p'_h}}^Y (\widehat{P}_Y))(1).$$

We can use here (1.26), describing the set of  $\widetilde{\beta}^p \stackrel{\text{def}}{=} (\pi_r s_{i_l} \cdots s_{i_{p+1}})(\alpha_{i_p})$  that may appear in (10.36) for a given reduced decomposition  $\pi_c = \pi_r s_{i_l} \cdots s_{i_1}$  and singular  $p$ . Setting  $\widetilde{\beta} = [\beta, \nu_\beta j]$ , it is as follows:

$$(10.43) \quad \begin{aligned} & \widetilde{\beta} \in \lambda(\pi_c^{-1}) \text{ and } q_\beta^{j+(\beta^\vee, c-u_c^{-1}(\rho_k))} = 1, \text{ where} \\ & \lambda(\pi_c^{-1}) = \{\widetilde{\alpha} = [\alpha, \nu_\alpha j] \in \widetilde{R}_+, -(c, \alpha^\vee) > j \geq 0\}. \end{aligned}$$

There are two immediate applications of this description:

- (1) the set  $\{\widetilde{\beta}^p\}$  does not depend on the choice of the reduced decomposition of  $\pi_c$ ;
- (2) the non-affine components of such  $\widetilde{\beta}$  cannot coincide for distinct  $\widetilde{\beta}$  for generic  $q$ .

**Corollary 10.7.** *Given  $\pi_c$ , let us assume that the singular roots in  $\lambda(\pi_c)$  are pairwise orthogonal; for instance, it holds for any  $\widehat{w}$  if  $\widetilde{R}^0$  is a direct sum of one dimensional root systems, respectively,  $\widetilde{W}^0$  is a commutative group. Then the module  $\widetilde{V}_c$  is  $Y$ -cyclic generated by  $\widetilde{E}_c$ .*

*Proof.* The corresponding Demazure operators for *singular* roots in (10.42) are pairwise commutative and it is possible to find a polynomial  $P^{p'_j}$  for each  $p'_j$  such that

$$D_{\widetilde{\alpha}^{p'_i}}^Y(\widehat{P}^{p'_j}) = \delta_{ij} \quad \text{for all } 1 \leq i, j \leq g.$$

One can use directly formula (10.36) here. Hence, the  $Y$ -span of  $\widetilde{E}_c$  contains  $\widetilde{E}_c^\dagger$  for  $g' = 1$  and  $c'$  as in (10.36). Then we can proceed by induction.  $\square$

It is not known how far the  $Y$ -modules  $\widetilde{V}_c$  are from cyclic in general. The strongest *criterion* we give in the paper is Proposition 10.12 below. Combined with Theorem 2.4, it shows that in quite a few cases  $\widetilde{V}_c$  are cyclic. It is not difficult to construct examples when  $\mathcal{V}_c$  are not  $Y$ -cyclic.



**Corollary 10.8.** (i) Given  $c \in B$  and a reduced decomposition of  $\pi_c$ , the polynomial  $\tilde{E}_c^\dagger \neq 0$  defined for the corresponding standard decomposition (maybe non-reduced) of  $\hat{w}' \in \tilde{\mathcal{B}}^0(\pi_c)$  is a  $Y$ -eigenvector for  $c' = \hat{w}'((0))$  if  $\tilde{E}_{c'}^\dagger = 0$  for all  $\pi_{c''} <_0 \pi_{c'}$ .

Any  $Y$ -quotient of  $\mathcal{V}_c$  contains at least one nonzero  $Y$ -eigenvector that is the image of  $\tilde{E}_{c'} \in \mathcal{V}_c$  for suitable  $c'$ ; one can take here  $c'$  such that  $\pi_{c'} \leq_0 \pi_c$  is minimal with respect to  $\leq_0$  considered only among  $\pi_{c'}$  with nonzero images of  $\tilde{E}_{c'}$ .

(ii) The kernel of  $\Psi_i$  in  $\mathcal{V}$  or its any finite dimensional  $Y$ -submodule consists only of  $Y_{\alpha_i}$ -eigenvectors provided that the action of  $\Psi_i$  is well defined at such vectors and  $t_i \neq \pm 1$ . Given  $c \in B$  and  $0 \leq i \leq n$ , the action of  $\Psi_i$  is well defined in the space  $\mathcal{V}_c$  unless  $s_i$  is singular in the product  $s_i \pi_c$ ; see (9.3).

Let  $s_i$  be singular and  $t_i \neq 1$ . Then  $(\alpha_i^\vee, c + d) \neq 0$  and  $s_i \pi_c \in \pi_B$ . Moreover, if  $0 \neq E_1 \in \tilde{V}_c$ , then at least one of the vectors  $E_1$  and  $E_2 = \tau_+(T_i)E_1$  is not a  $Y_{\alpha_i}$ -eigenvector.

*Proof.* Claim (i) follows from the fact that the  $Y$ -submodule generated by  $\tilde{E}_c$  is given in terms of  $\tilde{E}_{c'}$  for  $\pi_{c'} \leq_0 \pi_c$ .

The first claim in (ii) results from the formula

$$(\Psi_i)^2 = \frac{(t_i^{1/2} Y_{\alpha_i}^{-1} - t_i^{-1/2})(t_i^{1/2} Y_{\alpha_i} - t_i^{-1/2})}{(Y_{\alpha_i}^{-1} - 1)(Y_{\alpha_i} - 1)}.$$

Relation (10.38) can be naturally extended to the vectors  $[b, l]$  from  $[B, \mathbb{Z}]$ . Applying it to  $\alpha_i$ ,

$$(10.44) \quad (Y_{\alpha_i} - 1)E_2 = 2(t_i^{1/2} - t_i^{-1/2})E_1$$

if  $E$  is a  $Y_{\alpha_i}$ -eigenvector; the corresponding eigenvalue is 1. Respectively, if  $\tilde{E}$  is a  $Y$ -eigenvector, then

$$(Y_{\alpha_i}^{-1} - 1)E_1 = 2(t_i^{-1/2} - t_i^{1/2})E_2.$$

It gives the rest of (ii).  $\square$

**Corollary 10.9.** Let  $t_\nu \neq 1$  for all  $\nu$ . The condition  $\tilde{R}_+^0 \subset R_+$ , i.e., the absence of singular  $\alpha_i$  in all  $\pi_c$ , is necessary and sufficient for the polynomial representation  $\mathcal{V}$  to be  $Y$ -semisimple. In this case, any reduced chain originated at  $E_0 = 1$  consists of one-dimensional nonzero spaces  $\tilde{V}_c$ , even if some of the simple intertwiners  $\Psi_i$  become non-invertible

in this chain; all Macdonald polynomials  $E_c$  are well defined in such  $\mathcal{V}$  (and coincide with  $\tilde{E}_c$ ).

*Proof.* Use Corollary 10.8,(ii).

**10.5. Induced representations.** The  $Y$ -modules  $\tilde{V}_c$  are closely related to their counterparts for the  $Y$ -induced representations of  $\mathcal{H}^b$ .

Let us fix a weight  $\xi$ . In this section  $\tilde{R}^0$  is an arbitrary root subsystem from Section 4 satisfying:

$$(10.45) \quad q^{(\tilde{\alpha}, \xi + d)} = 1 \Rightarrow \tilde{\alpha} \in \tilde{R}^0.$$

The most natural choice (cf. (9.7)) is

$$(10.46) \quad \tilde{R}^0 \stackrel{\text{def}}{=} \{\tilde{\alpha} = [\alpha, \nu_\alpha j] \in \tilde{R} \mid q^{\nu_\alpha j + (\alpha, \xi)} = 1\}.$$

let  $\mathcal{I}_\xi$  be the  $\mathcal{H}^b$ -module induced from the corresponding representation of its subalgebra  $\mathbb{Q}_{q,t}[Y_a, a \in B]$ . By definition,  $\mathcal{I}_\xi$  is generated by the element  $v$  with the defining relations  $Y_a(v) = q^{(a, \xi)}v$  and is naturally isomorphic to  $\mathcal{H}_X^b = \langle T_i, 1 \leq i \leq n, X_b, b \in B \rangle$  due to the PBW-theorem for the pair of subalgebras  $\mathcal{H}_X^b$  and  $\mathbb{Q}_{q,t}[Y_b, b \in B]$ . It will be more convenient to use the identification of  $\mathcal{I}_\xi$  with  $\tau_+(\mathcal{H}_Y^b)$ :

$$\begin{aligned} \tau_+(\mathcal{H}_Y^b) &= \tau_+ \tau_-^{-1}(\mathcal{H}_Y^b) = \tau_- \sigma(\mathcal{H}_Y^b) \\ &= \langle \tau_+(T_i), 0 \leq i \leq n, \tau_+(\pi_r) \text{ for } b_r \in \Pi^b \rangle, \text{ where} \\ \mathcal{H}_Y^b &= \langle T_i, Y_b, b \in B \rangle = \langle T_i, 0 \leq i \leq n, \pi_r \text{ for } b_r \in \Pi^b \rangle. \end{aligned}$$

Here the automorphisms  $\tau_\pm, \sigma$  are used; see (5.8), (5.9), and (6.15). The isomorphism  $\mathcal{I}_\xi = \tau_+(\mathcal{H}_Y^b)$  (as vector spaces and as  $\tau_+(\mathcal{H}_Y^b)$ -modules) is based on the PBW-theorem for the pair of subalgebras  $\tau_- \sigma(\mathcal{H}_Y^b)$  and  $\mathbb{Q}_{q,t}[Y_b, b \in B] = \tau_- \sigma(\mathbb{Q}_{q,t}[X_b, b \in B])$ , which is the  $\tau_- \sigma$ -image of the standard PBW-theorem for  $\mathcal{H}_Y^b$  and  $\mathbb{Q}_{q,t}[X_b, b \in B]$ .

We introduce the  $\tilde{e}$ -elements in  $\mathcal{I}_\xi$ , counterparts of  $\tilde{E}$ -polynomials, as follows:

$$(10.47) \quad \tilde{e}_{\hat{w}} \stackrel{\text{def}}{=} \tilde{\Psi}_{\hat{w}}(v) \text{ for } \hat{w} \in \widehat{W}^b, \tilde{\Psi}_{\hat{w}} \text{ from Theorem 5.2.}$$

Recall that  $\tilde{\Psi}_{\hat{w}}$  is constructed by replacing  $\Psi_{i_p}$  with  $T_{i_p}$  for singular indices  $p$  in the product for  $\Psi_{\hat{w}}$  corresponding to a reduced decomposition of  $\hat{w} = \pi_r s_{i_l} \cdots s_{i_1}$ .

Similarly, one defines the spaces  $\mathcal{V}_{\hat{w}}^\xi, \tilde{V}_{\hat{w}}^\xi$ . The elements  $\tilde{e}_{\hat{w}}$  may depend on a choice of the reduced decompositions of  $\hat{w} \in \widehat{W}^b$  but they are

always nonzero. It follows from the calculation of the leading coefficient of  $\tilde{e}_{\hat{w}}$ , that is the coefficient of  $\tau_+(T_{\hat{w}})$ .

The space  $\mathcal{V}_{\hat{w}}^{\xi}$  is linearly generated by  $\tilde{e}_{\hat{w}'}$  for  $\hat{w}' \in \mathcal{B}^0(\hat{w})$ ;  $\tilde{V}_{\hat{w}}^{\xi} \subset \mathcal{V}_{\hat{w}}$ . It does not depend on the choice of the reduced decomposition and can be obtained by the same kind of limiting procedure as in the polynomial case. However, now the complete right Bruhat ordering must be used instead of its restriction to  $\pi_B$ .

If  $\xi = -\rho_k$ , then the images of  $\tilde{e}_{\hat{w}}$  under the  $\mathcal{H}^b$ -homomorphism  $\mathcal{I}_{\xi} \rightarrow \mathcal{V}$  sending  $v \mapsto 1$  become zero for  $\hat{w} \notin \pi_B$ , otherwise they are nonzero and proportional to  $\tilde{E}_b$  for  $\hat{w} = \pi_b$ . Respectively,  $\mathcal{V}_{\pi_c}^{\xi}$  and  $\tilde{V}_{\pi_c}^{\xi}$  map onto  $\mathcal{V}_c$  and  $\tilde{V}_c$ .

**The case of generic weights.** We will begin with generic  $\xi$  subject to (10.45). Practically, it means that the intertwiners can be singular, but all are invertible; cf. Proposition 8.11 from [L]. Then the normalized intertwiners  $F = \tau_- \sigma(G)$  can be used instead of the  $\Psi$ .

Note that it is exactly the case where there is a complete parallelism with Kauffman's axioms of *virtual links*; the *normalized* intertwiners and  $\{T\}$  provide the key example for Kauffman's axioms.

Thus one may switch from  $\tilde{e}_{\hat{w}}$  to  $\tilde{e}'_{\hat{w}} \stackrel{\text{def}}{=} F_{\hat{w}}(v)$ . Such  $\tilde{e}'_{\hat{w}}$  do not depend on the choice of the reduced decompositions and are proportional to  $\tilde{e}_{\hat{w}}$ ; this is due to using the normalized intertwiners.

**Proposition 10.10.** *Let  $\xi$  be generic subject to (10.45). Then*

- (i)  $\mathcal{V}_{\hat{w}}^{\xi} = \tilde{V}_{\hat{w}}^{\xi}$  and therefore the elements  $\tilde{e}_{\hat{w}'}$  for  $\hat{w}' \in \mathcal{B}^0(\hat{w})$  form a basis in the space  $\tilde{V}_{\hat{w}}^{\xi}$ ;
- (ii) the elements  $\{\tilde{e}_{\hat{w}'}^{\dagger}\}$ , defined for all standard decompositions of  $\hat{w}' \in \tilde{\mathcal{B}}^0(\hat{w})$ , can be linearly expressed in terms of  $\{\tilde{e}_{\hat{w}'}\}$ ;
- (iii) the expressions from (ii) combined with (10.42) and the quadratic relations for  $\{T\}$  give a complete description of the  $Y$ -action in  $\tilde{V}_{\hat{w}}^{\xi}$ .

□

The  $Y$ -structure of  $\tilde{V}_{\hat{w}}^{\xi}$  can be calculated for generic  $q, t$  in terms of the affine Hecke algebras associated with the root system  $\tilde{R}^0$ , provided that  $\xi$  is generic from (10.45).

The construction requires the set of positive roots  $\tilde{R}_+^0 \subset \tilde{R}^0$ , the corresponding simple roots  $\{\alpha_i^0\}$  in  $\tilde{R}_+^0$ , and the standard Bruhat ordering for  $\tilde{R}^0$ . Recall that the notation for the *standard Bruhat sets*

for  $\tilde{R}^0$  is  $\mathcal{B}(\tilde{u}; \tilde{R}_+^0)$  where  $\tilde{u} \in \tilde{W}^0 = \langle s_i^0 \rangle$ . Let  $\tilde{H}^0$  be the Hecke algebra defined for the system  $\tilde{R}^0$ ; it is generated by  $\{T_i^0\}$  satisfying the homogeneous Coxeter relations and the quadratic ones with  $t_{\alpha^0} = t_\alpha$  as  $\alpha^0 = [\alpha, \nu_{\alpha} j] \in \tilde{R}$ . The elements  $T_{\tilde{u}}^0 \in \mathcal{H}^0$  for  $\tilde{u} \in \tilde{W}^0$  are defined naturally and do not depend on the choice of the reduced decompositions of  $\tilde{u}$ :

$$\tilde{H}^0 = \oplus_{\tilde{u}} \mathbb{Q}_{q,t} T_{\tilde{u}}^0 \quad \text{for } \tilde{u} \in \tilde{W}^0.$$

Let  $B^0 \stackrel{\text{def}}{=} \{b \in B, (\xi, b) = 0\}$ ,  $\tilde{\mathcal{H}}_Y^0$  be the algebra generated by  $\mathbb{Q}_{q,t}[Y_b, b \in B^0]$  and the Hecke algebra  $\tilde{H}^0$ . We impose

$$(10.48) \quad Y_b T_i^0 = T_i^0 Y_{s_{\tilde{\alpha}}(b)} + (t_\alpha^{1/2} - t_\alpha^{-1/2}) \frac{Y_{s_{\tilde{\alpha}}(b)} - Y_b}{Y_{\tilde{\alpha}}^{-1} - 1} \quad \text{for } \tilde{\alpha} = \alpha_i^0;$$

to put it simply,  $T_i^0$  are  $\mathcal{T}_{\tilde{\alpha}_i^0}$  from (10.40).

The induced module  $\tilde{\mathcal{I}}^0$  is defined by setting  $Y_b(1) = 1$  and can be naturally identified with  $\tilde{H}^0$  (as  $\tilde{H}^0$ -modules).

**Proposition 10.11.** *Provided that  $q, t$  are not roots of unity and  $\xi$  is generic, the  $Y$ -module  $\tilde{V}_{\tilde{w}}^\xi$  is cyclic for  $\tilde{w} \in \tilde{W}^b$  if and only if the following holds in  $\tilde{\mathcal{I}}^0$  for  $\tilde{u} = \tilde{w}|_0 \in \tilde{W}^0$  and  $\mathbb{Q}_{q,t}^0[Y] \stackrel{\text{def}}{=} \mathbb{Q}_{q,t}[Y_b, b \in B^0]$ :*

$$(10.49) \quad \mathbb{Q}_{q,t}^0[Y] (T_{\tilde{u}}^0) = \oplus_{\tilde{u}'} \mathbb{Q}_{q,t} T_{\tilde{u}'}^0 \quad \text{for } \tilde{u}' \in \mathcal{B}(\tilde{u}; \tilde{R}_+^0),$$

where  $\mathcal{B}$  is the standard Bruhat ordering in  $\tilde{W}^0$ . For instance, relations (10.49) are satisfied if  $\tilde{u}$  is the element  $\tilde{u}_0$  of maximal length in  $\tilde{W}^0$ .

*Proof.* Using the intertwiners  $F$  and the polynomials  $\tilde{e}'_{\tilde{w}}$ , the calculation of Theorem 10.6 can be reduced to the case of  $\tilde{\mathcal{H}}_Y^0$  and  $\tilde{\mathcal{I}}^0$ .

Let us examine the case of  $\tilde{u} = \tilde{u}_0$ . The reduction to the degenerate affine Hecke algebra  $\tilde{\mathcal{H}}_y^0$  from [C1], Lemma 2.12 can be used (the analysis of irreducibility of  $\tilde{\mathcal{I}}^0$ ). Laurent polynomials in terms of  $\{Y_b\}$  are replaced by the usual polynomials in terms of  $\{y_b\}$ . The elements  $T_{\tilde{u}}^0$  become  $\tilde{u}$ ; the Demazure operators in formulas (10.48), (10.37) are replaced by the BGG operators  $d_{\tilde{\alpha}}^y = y_{\tilde{\alpha}}^{-1}(s_{\tilde{\alpha}} - 1)$ .

The degeneration of  $\tilde{\mathcal{I}}^0$  is isomorphic to the quotient  $\tilde{\mathcal{I}}_y^0$  of the standard representation of  $\tilde{\mathcal{H}}_y^0$  in the space of polynomials in terms of  $y_b$  for  $b \in B^0$  by the ideal generated by the  $\tilde{W}^0$ -invariant polynomials with zero constant term. It contains a unique  $y$ -eigenvector  $d$ , the *discriminant*, and  $\tilde{u}_0(d)$  is a linear generator of the one-dimensional space

$\tilde{\mathcal{I}}_y^0/(y_b, b \in B^0)$ ; therefore it is a  $y$ -cyclic vector in  $\tilde{\mathcal{I}}_y^0$  (Nakayama's lemma). It gives the degenerate version of (10.49) and results in the required claim.  $\square$

**Comment.** (i) A counterexample to (10.49) is  $\tilde{R}^0 = A_3$ ,  $\tilde{u} = (4231)$ . There are four elements in  $\mathcal{B}(\tilde{u})$  of length  $l(\tilde{u}) - 1$ , i.e., their number is greater than the rank. This leads to a contradiction (consider the degeneration above). One can use Proposition 2.3 to construct more general counterexamples.

(ii) Note that the construction we discuss is connected with the theory of Schubert polynomials upon the reduction we performed when proving the proposition. Using the BGG-operators for Schubert polynomials is similar to what we did. Generally, the combinatorics of *affine* Schubert manifolds has a lot in common with that of nonsymmetric Macdonald polynomials.  $\square$

**Arbitrary weights.** There is a natural extension of Proposition 10.11 to the case of arbitrary, *not only generic*,  $\xi$ . We impose (10.45);  $q, t$  will be not roots of unity. The sequence  $\beta_g, \beta_{g-1}, \dots, \beta_2, \beta_1$  of *simple, maybe coinciding* roots in  $\tilde{R}_+^0$  constructed in (4.62), (4.63) will be used; it is defined in terms of a given reduced decomposition of  $\hat{w} \in \hat{W}^b$ .

We continue using the Hecke algebra  $\tilde{H}^0$ . Given a reduced decomposition of  $\tilde{u} \in \tilde{W}^0$ , we will need the elements  $T_{\tilde{u}'}^{0\dagger} \in \tilde{H}^0$  for  $\tilde{u}' \in \mathcal{B}(\tilde{u}; \tilde{R}_+^0)$ , defined by crossing out the corresponding  $T_i^0$  in the corresponding product for  $T_{\tilde{u}}^0$ . In contrast to  $T_{\tilde{u}'}$ ,  $T_{\tilde{u}'}^{0\dagger}$  may depend on the choice of the reduced decomposition of  $\tilde{u}$  (unless the resulting decomposition of  $\tilde{u}'$  remains reduced). Respectively,  $\mathcal{B}^\dagger(\tilde{u}; \tilde{R}^0)$  is the set of such decompositions, called *standard* in (10.34).

**Proposition 10.12.** *Let all (simple) singular reflections in the decomposition of  $\hat{w}$  belong to a disjoint union  $\{L_j, 1 \leq j \leq r\}$  of consecutive segments (connected portions) of a given reduced decomposition of  $\hat{w}$ ;  $\cup_j L_j$  may be smaller than  $\lambda(\hat{w})$ , there can be gaps between  $L_j, L_{j+1}$ . Following Proposition 4.5, we obtain the reduced decomposition  $\hat{w}|_0 = s_{\beta_g} \cdots s_{\beta_1}$  for simple  $\beta_j \in \tilde{R}^0$ . In terms of  $L_j$ ,  $\hat{w}|_0 = \tilde{u}_r \cdots \tilde{u}_1$ , where  $\tilde{u}_j = s_{\beta_{b_j}} \cdots s_{\beta_{a_j}}$  is the part of the decomposition of  $\hat{w}|_0$  corresponding to  $L_j$  for  $a^1 \leq b^1 < a^2 \leq b^2 \dots$  determining this partition.*

We impose (10.49) for  $\tilde{u} = \hat{w}|_0$ , however, do not assume  $\xi$  to be generic. Then  $Y$ -module  $\tilde{V}_{\tilde{w}}^{\xi}$  is cyclic under the assumptions:

(i)  $q^{(\tilde{\alpha}, \xi+d)} \neq t_{\tilde{\alpha}}^{\pm 1}$  for  $\tilde{\alpha} = \tilde{\alpha}^m \in \lambda(\hat{w})$  corresponding to the (simple) reflections  $s_{i_m} \in \cup_j L_j$  for  $\hat{w} = \pi_r s_{i_l} \cdots s_{i_1}$ , i.e., non-singular intertwiners  $\tilde{\Psi}_{i_m}$  in  $\tilde{\Psi}_{\hat{w}}(v)$  are all invertible for such  $s_{i_m}$ ;

(ii) if  $T_{\tilde{u}'}^{0\dagger} \in \tilde{H}^0$  are linearly dependent for some  $\tilde{u}' \in \tilde{U}' \subset \mathcal{B}^{\dagger}(\tilde{u}; \tilde{R}_+^0)$  then the singular reflections removed to obtain the standard decompositions of these  $\tilde{u}' \in \tilde{U}'$  must be all from one  $L_j$ , i.e., can be obtained from the same  $\tilde{u}_j$ .  $\square$

Note that (ii) holds if  $\tilde{u}' \neq \tilde{u}''$  for elements  $\tilde{u}', \tilde{u}'' \in \mathcal{B}(\tilde{u}; \tilde{R}_+^0)$  that come from different  $L_j$ , however (ii) is of more general nature. For instance, if all singular reflections form a connected segment in the reduced decomposition of  $\hat{w}$  then (i,ii) hold.

Assuming that there exists  $\hat{v} \in \widehat{W}^b$  sending *simple* roots of  $\tilde{R}_+^0$  to *simple* roots of  $\tilde{R}_+$ , the element

$$(10.50) \quad \hat{w} = \hat{v}\tilde{u}_0 = s_{\alpha(g)} \cdots s_{\alpha(1)} \hat{v} \quad \text{for} \quad \hat{v}(\{\beta_j\}) = \{\alpha(j)\}, \quad 1 \leq j \leq g,$$

for the longest  $\tilde{u}_0 \in \tilde{W}^0$  satisfies all conditions of the proposition. The corresponding module  $\tilde{V}_{\tilde{w}}^{\xi}$  is  $Y$ -cyclic. Indeed,  $\hat{v}(\beta) > 0$  for any  $\beta \in \tilde{R}_+^0$  by construction, therefore  $\lambda(\hat{v})$  contains no singular roots. Any elements  $\hat{v}'\tilde{u}_0$  such that  $l(\hat{v}'\tilde{u}_0) = l(\hat{v}'\hat{v}^{-1}) + l(\hat{v}\tilde{u}_0)$  can be taken too, since  $\lambda(hv'\tilde{u}_0) \setminus \lambda(\hat{v}\tilde{u}_0)$  cannot contain roots from  $\tilde{R}_+^0$  (they are all already in  $\lambda(\hat{v}\tilde{u}_0)$ ). We come to the following corollary.

**Corollary 10.13.** *Assuming that  $q, t$  are not roots of unity and an element  $\hat{v} \in \widehat{W}^b$  exists sending simple roots of  $\tilde{R}_+^0$  to simple roots of  $\tilde{R}_+$ , one can find  $\hat{w}$  such that the modules  $\tilde{V}_{\hat{w}'}^{\xi}$  are  $Y$ -cyclic when  $l(\hat{w}') = l(\hat{w}'\hat{w}^{-1}) + l(\hat{w})$ , i.e. when  $\hat{w}'$  are sufficiently big.  $\square$*

Note that if  $\tilde{V}_{\tilde{w}}^{\xi}$  is cyclic for  $\xi = -\rho_k$  and  $\hat{w} = \pi_c$ , then  $\tilde{V}_c$  is cyclic in the polynomial representation; thus we can use Corollary 10.13 and Proposition 10.12 to study  $\mathcal{V}$ . For instance, one can take  $\hat{v}\tilde{u}_0$  for  $\hat{v}$  from (10.50) provided that it is in the form  $\pi_c$  for certain  $c$ . The latter means that  $\hat{v} = \pi_b$  and the roots from  $\tilde{R}_+^0$  have negative nonaffine components.

## 11. THE RADICAL

The technique of intertwiners is expected to help in decomposing  $\mathcal{V}$  in terms of the irreducible constituents. The first step in this direction is finding **singular**  $q, t_\nu$  making the *radical* of the polynomial representation  $\mathcal{V}$  nonzero. In this section  $q$  is generic, not a root of unity, so the problem is in finding singular  $t_\nu$  in terms of  $q$ . The radical is an  $\mathcal{H}^\flat$ -submodule defined for the “evaluation pairing” in  $\mathcal{V}$ . There are several cases when the radical is zero but  $\mathcal{V}$  is reducible (this never happens in the rational case and for simply-laced root systems); the technique of intertwiners makes it possible to describe these cases.

Actually using the intertwiners alone is essentially sufficient to describe all cases when  $\mathcal{V}$  becomes reducible (for instance, it is possible in the simply-laced case) without any reference to the radical. However, the approach via the radical significantly simplifies the combinatorics involved.

In the  $A - D - E$ -case, the answer is simple to formulate. It follows from Theorem 11.8 (see also (7.14)) and the Zigzag Lemma 12.4 below.

**Theorem 11.1.** *Let  $t = t_{\text{sht}}$ ,  $\zeta_i$  be primitive  $(m_i+1)$ th roots of unity for the classical exponents  $m_i$  in the simply-laced case; see (7.13). When  $q$  is not a root of unity, the radical is nonzero if and only if*

$$t = q^{-l - \frac{j}{m_i+1}} \zeta_i^{j'} \text{ as } 1 \leq i \leq n, 0 \leq j, j' \leq m_i, j + j' > 0, l \in 1 + \mathbb{Z}_+.$$

*Moreover, the radical is nonzero if and only if  $\mathcal{V}$  is reducible.  $\square$*

**11.1. Basic properties.** Recall that the *evaluation pairing* from (6.32) is as follows:

$$(11.1) \quad \{f, g\} = \{L_{\iota(f)}(g(X))\} = \{L_{\iota(f)}(g(X))\}(q^{-\rho_k}) \text{ for } f, g \in \mathcal{V},$$

$$\iota(X_b) = X_{-b} = X_b^{-1}, \iota(z) = z \text{ for } z \in \mathbb{Q}_{q,t}.$$

It induces the  $\mathbb{Q}_{q,t}$ -linear anti-involution  $\phi$  of  $\mathcal{H}^\flat$  from (5.11):

$$(11.2) \quad \phi \stackrel{\text{def}}{=} \varepsilon \star = \star \varepsilon : X_b \mapsto Y_b^{-1}, T_i \mapsto T_i \text{ } (1 \leq i \leq n).$$

**Lemma 11.2.** *For arbitrary nonzero  $q, t_{\text{sht}}, t_{\text{lng}}$ ,*

$$(11.3) \quad \{f, g\} = \{g, f\} \text{ and } \{H(f), g\} = \{f, H^\phi(g)\}, H \in \mathcal{H}^\flat.$$

*The quotient  $\mathcal{V}'$  of  $\mathcal{V}$  by the radical  $\text{Rad} \stackrel{\text{def}}{=} \text{Rad}\{, \}$  of the pairing  $\{, \}$  is an  $\mathcal{H}^\flat$ -module such that*

- (a) all  $Y$ -eigenspaces of  $\mathcal{V}'$  are zero or one-dimensional,
- (b)  $E(q^{-\rho_k}) \neq 0$  if the image  $E'$  of  $E$  in  $\mathcal{V}'$  is a nonzero  $Y$ -eigenvector.

*Proof.* Formulas (11.3) are from Theorem 2.2 of [C6]. See also [C7], Corollary 5.4. We recall the argument from [C9]. Since  $\text{Rad}\{, \}$  is a submodule, the form  $\{, \}$  is well defined and nondegenerate on  $\mathcal{V}'$ . For any pullback  $E \in \mathcal{V}$  of  $E' \in \mathcal{V}'$ ,  $E(q^{-\rho_k}) = \{1, E\} = \{1', E'\}$ . If  $E'$  is a  $Y$ -eigenvector in  $\mathcal{V}'$  and  $E(q^{-\rho_k})$  vanishes, then

$$\{\mathcal{H}_Y^b(1'), Q_{q,t}[Y_b](E')\} = 0 = \{\mathcal{V} \cdot \mathcal{H}_Y^b(1'), E'\}.$$

Therefore  $\{\mathcal{V}', E'\} = 0$ , which is impossible.  $\square$

The following lemma follows directly from the definition of the radical.

**Lemma 11.3.** *The radical  $\text{Rad}$  is the greatest  $\mathcal{H}^b$ -submodule in the kernel of the map  $f \mapsto \{1, f\} = f(q^{-\rho_k})$ . It is also the intersection of the kernels of evaluation maps  $f \mapsto f_c^\vee(e) = \{e, f\}$  for all  $e \in \mathcal{V}(-c_\#)^\infty$ :*

$$(11.4) \quad \xi_c : \mathcal{V} \ni f \mapsto f_c^\vee \in (\mathcal{V}(-c_\#)^\infty)^\vee \stackrel{\text{def}}{=} \text{Hom}(\mathcal{V}(-c_\#), \mathbb{Q}_{q,t}),$$

where the later linear space is supplied with the following natural action of  $\mathbb{Q}_{q,t}[X_b]$ :  $X_b(f_c^\vee(e)) = f_c^\vee(Y_b^{-1}(e))$  for  $e \in \mathcal{V}(-c_\#)^\infty$ . Here the maps  $\xi_c$  are  $X$ -homomorphisms and their kernels are ideals in  $\mathcal{V}$ .  $\square$

Using the spaces  $\tilde{V}_c$  from Section 9.2, defined for reduced decompositions of  $\pi_c$ , we can improve this statement. Namely,  $f \in \text{Rad}$  if and only if the  $X$ -homomorphisms

$$(11.5) \quad \zeta_c : \mathcal{V} \ni f \mapsto f_c^\vee \in \tilde{V}_c^\vee = \text{Hom}(\tilde{V}_c, \mathbb{Q}_{q,t})$$

are all zero for  $f \mapsto f_c^\vee = \{\tilde{E}, f\}$ , where  $\tilde{E} \in \tilde{V}_c$ .

It is important to know how the zero-value condition  $\zeta_c(\tilde{E}) = 0$  is transformed when  $c$  changes. Let us assume that  $(\alpha_i^\vee, c) > 0$  and that  $\Psi_i^c$  is infinity. i.e., satisfies (9.3). Then the zero-value condition for the next  $\tilde{V}_{s_i((c))} = \tilde{V}_c + \tau_+(T_i)\tilde{V}_c$  reads as follows:

$$\zeta_c(\tau_+(T_i)(\tilde{E})) = 0 \text{ and } \zeta_c(\tilde{E}) = 0.$$

Here we use (5.12):

$$\phi(\tau_+(T_i)) = (\star \cdot \tau_+ \cdot \eta)(T_i) = \tau_+(T_i) \text{ for } n \geq i \geq 0.$$

Recall that  $\tau_+(T_i) = T_i$  for  $i > 0$  and  $\tau_+(T_0) = X_0^{-1}T_0^{-1}$  for  $X_0 = qX_{\vartheta}^{-1}$ .



The cases  $(a, c)$  from the definition of  $\tilde{V}$ -spaces can be readily considered using formulas (6.39) for the action of the  $\phi$ -images of the simple intertwiners acting on polynomials.

Combining Lemma 11.3 and Lemma 11.2, we come to the following lemma.

**Lemma 11.4.** *(i) A  $Y$ -eigenvector  $E$  belongs to  $\text{Rad}$  if and only if  $E(q^{-\rho_k}) = 0$ . It holds true if the later vanishing condition is replaced by  $E(q^{-c_\sharp}) = 0$  provided that  $\Psi_{\pi_c}$  evaluated at  $q^{-\rho_k}$  is an invertible element in the subalgebra  $\mathcal{H}_X^b$  of  $\mathcal{H}^b$ , which is generated by  $X_a (a \in B)$ , and  $T_i (i > 0)$ . For instance,  $c = \pi_r((0)) = \omega_r$  can be taken instead of  $c = 0$  for  $r \in O$ ,  $\omega_r \in B$ ; similarly,  $c = s_0((0)) = \vartheta$  can be taken instead of 0 provided that  $qt_0^{(\rho, \vartheta)} \neq t^{\pm 1}$ .*

*(ii) The equality  $E(q^{-\rho_k}) = 0$  automatically results in the equalities*

$$(11.6) \quad E(q^{-b^\circ_\sharp}) = 0 \quad \text{for } b^\circ \in B^* \stackrel{\text{def}}{=} \{b = b^\circ \in B \mid E_{b^\circ}(q^{-\rho_k}) \neq 0\},$$

where  $E_{b^\circ}$  is the Macdonald polynomial for primary  $b^\circ$  defined in (9.2).

*(iii) Expanding  $E = \sum_{\mathbf{m}} C_{\mathbf{m}} X_1^{m_1} \cdots X_n^{m_n}$ ,  $C_{\mathbf{m}} \neq 0$  for  $E$  from (i) and setting  $X_i = X_{b_i}$ , the degree  $\text{Deg}(E)$  is defined as*

$$\text{Deg}(E) \stackrel{\text{def}}{=} \text{Max}_{\mathbf{m}} \{m_1 + \dots + m_n\} - \sum_i \text{Min}_{\mathbf{m}} \{m_i\}.$$

Then  $\text{Deg}(E)^n$  is no smaller than the cardinality of the intersection  $\cap_{i=1}^n \{c^i + B_\sharp^* \cap B_\sharp^*\}$ , where  $B_\sharp^* = B^* \cap B_\sharp$  and the translations  $c^i(E)$  of  $E$  are assumed to have finitely many common zeros; such set  $\{c^i\}$  always exists in  $B_+$ , more generally, in  $u(B_+)$  for an arbitrary  $u \in W$ .

*Proof.* The first claim follows from Lemma 11.2. Let us check that  $E \in \text{Rad}$  if and only if  $\{A, E\} = 0$  for one invertible  $A \in \mathcal{H}_X^b$ . Indeed,

$$\begin{aligned} \{A \mathbf{H} Y_a(1), \mathbb{Q}_{q,t}[Y](E)\} = 0 &\Rightarrow \{\mathbb{Q}_{q,t}[X] A \mathbf{H} Y_a(1), E\} = 0 \\ &\Rightarrow \{\mathcal{H}_X^b Y_a(1), E\} = 0 \quad \text{for any } a \in B, \end{aligned}$$

where the nonaffine Hecke algebra  $\mathbf{H} \subset \mathcal{H}^b$  is used.

Coming to (iii), an arbitrary monomial  $X_b$  in the decomposition of  $E$  can be represented as a linear combination  $\sum_i \text{Coeff}_i c^i(E)$  for proper coefficients  $\text{Coeff}_i \neq 0$  and sufficiently large number of translations  $c^i$ . The translations  $c^i(E)$  cannot have common zeros in  $C^n$  here, because otherwise  $X_b$  would have such a zero. Moreover,  $\text{Deg}(E)^n$  is exactly the number of common zeros, counted with multiplicities, for  $n$  generic

(transversal) translations  $c^i(E)$ . Here  $c^i$  can be taken from  $B$  or from any  $u(B_+)$ .  $\square$

As an immediate application, we conclude that the set  $B^*$  must be always smaller than the whole  $B$  if the radical is nonzero. Setting  $u = \text{id}$  and using the previous lemma, we obtain that the radical always contains *symmetric* Macdonald polynomials.

**Lemma 11.5.** *Let us suppose that the radical  $\text{Rad}$  is nonzero. Then, given  $C_i > 0$  ( $i > 0$ ), it contains at least one Macdonald polynomial  $E_{b^\circ}$  for primary  $b^\circ = b_+^\circ$  satisfying  $(b^\circ, \alpha_i) > C_i$ ; see (9.2). In particular, the corresponding symmetric Macdonald polynomial  $P_{b_-^\circ} = \mathbf{P}(E_{b_+^\circ})$  is well defined for  $b_-^\circ = w_0(b_+^\circ)$  (see (6.14)) and belongs to the radical.*

*Proof.* Generally, if  $E_{b_+}$  is well-defined, then so is  $P_{b_-}$ , and

$$\Pi_R \{E_b, 1\} = \{P_{b_-}, 1\} = P_{b_-}(q^{-\rho_k}) \text{ for the Poincaré polynomial } \Pi_R.$$

Here we apply the symmetrization  $\mathbf{P}$  to 1, move  $\mathbf{P}$  to  $E_b$  and use (6.14);  $\mathbf{P}(1) = \Pi_R$  due to (7.15). Moreover, if  $\Pi_R \neq 0$  then

$$(11.7) \quad \begin{aligned} E_b \in \text{Rad} &\Rightarrow P_{b_-} \in \text{Rad} \Rightarrow \{P_{b_-}, 1\} = 0, \\ \{P_{b_-}, 1\} = 0 &\Rightarrow \{E_b, 1\} = 0 \Rightarrow E_b \in \text{Rad}. \end{aligned}$$

Then we use Lemma 11.4,(iii).  $\square$

**11.2. Non-negative  $k$ .** As another application, we come to the following theorem generalizing the description of singular  $k$  from [O2]; it is equivalent to the description below from Theorem 11.8 in terms of the shift-operators.

**Theorem 11.6.** *Assuming that  $q$  is generic, the radical vanishes if and only if  $E_{b^\circ}(q^{-\rho_k}) \neq 0$  for all sufficiently big primary  $b^\circ$ , i.e., if the product in the right-hand side of (6.27) is nonzero for all  $b \in B$  with sufficiently large  $(b, \alpha_i)$  for  $i > 0$ . Here  $t_\nu \neq 0$  are arbitrary.  $\square$*

We are going to check the irreducibility of  $\mathcal{V}$  in the case of non-negative  $k$  in this section; it automatically gives that  $\text{Rad} = \{0\}$ . One can of course use Theorem 11.6 to check that  $\text{Rad} = \{0\}$  for non-negative  $k$ .

**Proposition 11.7.** *Let us assume that  $\Re k_{\text{lng}} \geq 0$  and  $\Re k_{\text{sh}} \geq 0$  for  $q > 1$ . Then all elements  $\Psi_i^c, c \in B$  from (6.19), they belong to  $\mathbf{H}$ , are invertible, the polynomial representation  $\mathcal{V}$  is  $Y$ -semisimple with simple  $Y$ -spectrum and, moreover, irreducible.*

*Proof.* Setting  $\alpha^\vee = \sum_i m_i \alpha_i^\vee$  for  $\alpha \in R$ , let

$$\text{sht}(\alpha) = \sum_{\alpha_i = \text{sht}} m_i, \quad \text{lng}(\alpha) = \sum_{\alpha_i = \text{lng}} m_i.$$

Then  $(\alpha^\vee, \rho_k) = \text{sht}(\alpha)k_{\text{sht}} + \text{lng}(\alpha)k_{\text{lng}}$ . In the simply-laced case we set  $\text{lng}(\alpha) = 0$ , i.e., treat all roots as short. Note that  $\text{sht}(\alpha) > 0$  for short  $\alpha$  and  $\text{lng}(\alpha) > 0$  for long  $\alpha$ . Indeed, the intersections of  $R$  with the lattices  $\oplus_{\nu_i = \nu} \mathbb{Z}\alpha_i$  for fixed  $\nu = \nu_{\text{sht}}$ ,  $\nu = \nu_{\text{lng}}$  contain only roots  $\alpha$  of the same  $\nu_\alpha = \nu$ . These root subsystems of  $R$  are respectively the sets of  $\alpha$  such that  $\text{lng}(\alpha) = 0$  and  $\text{sht}(\alpha) = 0$ .

Recall the conditions that are necessary and sufficient for the intertwiner  $\Psi_i^c$  to be proportional to

$$(a) \tau_+(T_i) + t_i^{-1/2}, \quad (b) \tau_+(T_i) + \infty, \quad (c) \tau_+(T_i) - t_i^{1/2};$$

they are as follows:

$$(11.8) \quad q_\alpha^{-k_\alpha + (\tilde{\alpha}^\vee, c_- + d) - (\alpha^\vee, \rho_k)} = 1 \quad \text{for } (a) : \tau_+(T_i) + t_i^{-1/2},$$

$$(11.9) \quad q_\alpha^{(\tilde{\alpha}^\vee, c_- + d) - (\alpha^\vee, \rho_k)} = 1 \quad \text{for } (b) : \tau_+(T_i) + \infty,$$

$$(11.10) \quad q_\alpha^{k_\alpha + (\tilde{\alpha}^\vee, c_- + d) - (\alpha^\vee, \rho_k)} = 1 \quad \text{for } (c) : \tau_+(T_i) - t_i^{1/2},$$

where we set

$$\tilde{\alpha} = u_c(\alpha_i), \quad \alpha = u_c(\alpha_i) \quad \text{for } i > 0, \quad \text{and } \alpha = u_c(-\vartheta) \quad \text{for } i = 0.$$

Here  $\nu_\alpha = \nu_i$ ,  $\alpha < 0$ , and  $(\tilde{\alpha}, c_- + d) = (\alpha_i, c + d) > 0$ . See (1.39). In case (b), the intertwiner  $\Psi_i^c$  is not well defined. We will call it *infinity* and the corresponding  $s_i$  *singular* following the previous sections.

Due to the positivity assumptions,

$$-\Re(k_\alpha + (\alpha^\vee, \rho_k)) \geq 0 \quad \text{and} \quad \Re(-k_\alpha + (\tilde{\alpha}^\vee, c_- + d) - (\alpha^\vee, \rho_k)) > 0.$$

Thus (11.8)-(11.10) never take place, all intertwiners are invertible, and all  $E$ -polynomials are well defined. This approach actually allows  $\Re k$  to be “small” negative.

Recall that the eigenvalues are  $\{q^{-c_\#} = q^{u_c^{-1}(\rho_k - c_-)}\}$ . Generally,

$$(11.11) \quad q^{u_c^{-1}(\rho_k - c_-)} \neq q^{u_b^{-1}(\rho_k - b_-)} \quad \text{for } c \neq b \in B \quad \text{if} \\ \Re(\rho_k - c_-) \in \{z \in \mathbb{R}^n, \Re(\alpha_i^\vee, z) > 0\} \ni \Re(\rho_k - b_-),$$

for instance, for sufficiently large  $(\alpha_i^\vee, -c_-)$  for all  $i > 0$ . Indeed, then  $u_c = u_b$  and  $c_- = b_-$ . Due to  $\Re k \geq 0$ , the inequalities  $\Re(\alpha_i^\vee, z) \geq 0$  are sufficient in (11.11) and the latter holds for any  $c, b \in B$ .

Concerning the irreducibility, if  $\mathcal{V}' \subset \mathcal{V}$  is an  $\mathcal{H}^b$ -submodule then it contains at least one eigenvector  $v$  corresponding to  $q^{-c\sharp}$ . Applying the intertwiners, it must contain all eigenvalues (including the simple ones for sufficiently big  $c$  if  $\Re k$  is allowed to be small negative). Therefore it contains at least one  $E$ -polynomial. However then the intertwiners will make all of them in  $\mathcal{V}'$ .  $\square$

**Comment.** The method above gives the irreducibility of  $\mathcal{V}$  as  $\Re k > -1/h$  for the Coxeter number  $h = 1 + (\rho, \vartheta)$  in the simply-laced case; the inequalities are somewhat more involved for  $B, C, F, G$ . Here one can also use an analytic variant of the inner product (6.8) in  $\mathcal{V}$ . Let us outline an approach via the roots of unity assuming that  $k_{\text{lng}}, k_{\text{sht}}$  are rational numbers, which is sufficient for the analysis of the irreducibility.

Following [C6], (6.12)-(6.14),  $\mathcal{V}$  can be supplied with a  $\star$ -invariant inner product, where  $q^N = 1$  for sufficiently large  $N$  (also satisfying certain divisibility conditions). Under  $k > -1/h$  and the corresponding conditions in the non-simply-laced case, this inner product is nonzero and is *positive definite* on  $W$ -invariant polynomials. Then we use that reducibility of  $\mathcal{V}$  implies the reducibility of  $\mathcal{V}^W$  under the action of the subalgebra of invariants of  $\mathcal{H}^b$  (for generic  $q$ ) and that  $\mathcal{V}$  is in the category  $\mathcal{O}_Y$ .  $\square$

**11.3. Using affine exponents.** Continuing to assume that  $q$  is in a general position (see the exact condition in the theorem below), we will examine when  $Rad$  is zero via the shift operator. As a matter of fact, the answer has been already obtained (Theorem 11.6). However, using the shift operator has certain technical advantages in the  $q, t$ -case and generalizes Opdam's analysis of the rational case (where it is the only known approach). Note that Theorem 11.6 has no *differential* counterparts, rational or trigonometric.

The set of *singular*  $k$  with  $Rad \neq \{0\}$  is given by some algebraic equalities. Since  $q$  is generic (only roots of unity must be avoided), real  $q > 1$  are sufficient to consider.

We will apply formulas (7.11), (7.12) :

$$(11.12) \quad \Pi_{\tilde{R}} P_{b+\rho}^{q,t}(q^{-\rho_{k+1}}) = q^{\{\cdots\}} \left( \prod_{\alpha \in R_+} 1 - q_{\alpha}^{k_{\alpha} + (\alpha^{\vee}, \rho_k - b)} \right) P_b^{q,t}(q^{-\rho_k}).$$

Recall the notation

$$(tq^j)^{(\cdot, \rho^\vee)} = \prod_{\nu} (t_{\nu} q_{\nu}^j)^{(\cdot, \rho_{\nu}^\vee)};$$

see (7.9) for the definition of  $\Pi_{\tilde{R}}$ .

**Main Theorem 11.8.** (i) We assume that  $q \neq 0$  is not a root of unity. For the Poincaré polynomial  $\Pi_R$  from (7.15),  $\text{Rad} = \text{Rad}\{ , \}$  is zero if and only if

$$(11.13) \quad \Pi_R^{-1} \prod_{l=0}^{\infty} \text{rad}(q, tq^l) \neq 0 \quad \text{for} \quad \text{rad}(q, t) \stackrel{\text{def}}{=} \Pi_{\tilde{R}}$$

$$= \prod_{\alpha \in R_+} \left( (1 - t_{\alpha}(tq)^{(\alpha, \rho^\vee)}) \prod_{j=1}^{(\alpha^\vee, \rho)} \frac{(1 - q_{\alpha}^{j-1} t_{\alpha} t^{(\alpha, \rho^\vee)})}{(1 - q_{\alpha}^{j-1} t^{(\alpha, \rho^\vee)})} \right).$$

(ii) Provided that  $\rho \in B$  and  $\Pi_R \neq 0$ , the zeros of  $\Pi_{\tilde{R}}$  described explicitly in formula (7.14) in the cases  $\tilde{A}, \tilde{D}, \tilde{E}$ ,

$$(7.22), (7.24) \text{ for } \tilde{B}, \quad \text{formula (8.31) for } \tilde{C},$$

$$(8.33), (8.34) \text{ for } \tilde{G}, \text{ and } (8.36), (8.37) \text{ for } \tilde{F}$$

constitute the set of all  $q, t$  such that the  $t$ -discriminant  $\mathcal{X}^t$  belongs to  $\text{Rad}$ ; see formula (7.1). Here  $q \neq 0$  can be arbitrary, possibly a root of unity.

(iii) If  $q$  is not a root of unity ( $B$  can be arbitrary), then the zeros of  $\Pi_{\tilde{R}}/\Pi_R$  from (i) are those listed in (ii) and their positive translations. The latter are obtained from the zeros in (ii) with  $k_{\nu}$  replaced by all  $k'_{\nu} \in k_{\nu} + \mathbb{Z}_+$  disregarding the binomials that do contain the factors  $q^j$  with  $j > 0$ . Moreover, if

$$(11.14) \quad \{q^a \prod_{\nu} t_{\nu}^{b_{\nu}} = 1 \text{ for } a, b \in \mathbb{Z}\} \Rightarrow \{a + \sum_{\nu} \nu k_{\nu} b_{\nu} = 0\}, \quad \nu \in \nu_R,$$

then only the rational exponents, described in Theorem 8.1, are sufficient to consider here and in (ii).

*Proof.* If the radical is nonzero then it contains symmetric Macdonald polynomials  $P_b^{q,t}$  with sufficiently large negative  $b = b_-$  due to Lemma 11.5. Recall that  $P_b^{q,t} \in \text{Rad} \Leftrightarrow P_b^{q,t}(q^{-\rho_k}) = 0$  provided that  $\Pi_R \neq 0$ ; see (11.7). Let us assume that  $\Pi_R \neq 0$  in the following reasoning.

We apply consecutively the shift operators changing

$$b \mapsto b^{(m)} = b + m\rho, \quad k_{\nu} \mapsto (k + m)_{\nu} = k_{\nu} + m, \quad t \mapsto t^{(m)} = tq^m,$$

to make  $\Re(k+m+1)_\nu \geq 0$ . The radical is trivial for  $k+m+1$  thanks to Proposition 11.7 and  $P_{b^{(m+1)}}^{q,t^{(m+1)}}(q^{-\rho_{k+m+1}}) \neq 0$ . This step is similar to the approach of [O2, DJO] in the rational case, although the relation between the evaluation pairing and the shift operators is different in the rational and the  $q, t$ -cases.

Due to Theorem 7.3, see also (11.12),

$$\begin{aligned} \{\overline{\mathcal{Y}}^{t^{(m)}} \mathcal{X}^{t^{(m)}}\} P_{b^{(m+1)}}^{q,t^{(m+1)}}(q^{-\rho_{k+m+1}}) &= \overline{g}^{q,t^{(m)}}(b) P_{b^{(m)}}^{q,t^{(m)}}(q^{-\rho_{k+m}}), \\ \overline{g}^{q,t^{(m)}}(b) &= \prod_{\alpha \in R_+} ((t_\alpha q_\alpha^m)^{-1} X_\alpha(q^{(b-\rho_k)/2}) - X_\alpha(q^{(\rho_k-b)/2})), \end{aligned}$$

where  $\{\overline{\mathcal{Y}}^{t^{(m)}} \mathcal{X}^{t^{(m)}}\}$  is  $\text{rad}(q, tq^m)$  up to a power of  $q$ . Here  $\overline{g}^{q,t^{(m)}}(b)$  can be supposed nonzero because  $b$  is assumed sufficiently large negative.

We note that there are no restrictions concerning  $q, t$ , when using the shift operator (although the relation  $\Pi_R \neq 0$  is necessary for the applications to the radical). Also the above formula holds for *arbitrary* symmetric  $\mathbb{Q}_{q,t}[Y]^W$ -eigenfunctions  $P \in \mathcal{V}$  with the  $Y$ -weight coinciding with the weight of  $P_b$  ( $b = b_- \in B_-$ ) and such that  $P = \sum_{a \in W(b_-)} X_a$  modulo  $X_c$  for  $c_- \succ b_-$ ; see (6.12), (1.35).

Thus all partial products

$$\text{rad}(q, t) \text{rad}(q, tq) \cdots \text{rad}(q, tq^l) \quad \text{for } l \geq m$$

must vanish.

Without imposing  $\Pi_R \neq 0$  and using the shift-operator, one can use directly Theorem 11.6, which claims that the radical vanishes if and only if  $E_b(q^{-\rho_k}) \neq 0$  for sufficiently large  $b$ . Using the evaluation formula (6.27) as  $\lambda(\pi_b) \rightarrow \{[-\alpha, \nu_\alpha j], \alpha \in \mathbb{R}_+, j > 0\}$  we come exactly to (11.13).

Claim (ii). Thanks to Theorem 7.2,  $\text{rad}(q, t) = \Pi_{\tilde{R}} = 0$  if and only if  $\{\mathcal{X}^t, \mathcal{X}^t\} = 0$ ; cf. [O2], [DJO]. We need to check that  $\{\mathcal{X}^t, P\} = 0$  for an arbitrary polynomial  $P \in \mathcal{V}$ . Applying the  $t$ -anti-symmetrization (here  $\Pi_R = \mathbf{P}(1) \neq 0$  is needed), it suffices to assume that  $P$  is  $t$ -antisymmetric. Due to the inequalities  $t_\nu \neq -1$  (which follow from  $\Pi_R \neq 0$ ), it must be then divisible by  $\mathcal{X}^t$ , which gives the required.

The other claims from (ii,iii) follow from the definition and calculations of the affine and rational exponents performed in the formulas listed in (ii), (iii).  $\square$

**Comment.** Using the formula for  $\overline{\mathcal{Y}}^t(\mathcal{X}^t)$  with  $k'_\nu \in k_\nu + \mathbb{Z}_+$  in (ii,iii) instead of  $k$  is essentially equivalent to direct using the evaluation formula for the  $E$ -polynomials or  $P$ -polynomials. The approach without the shift-operator is more general, since it is not necessary to assume that  $\Pi_R \neq 0$ . On the other hand, using  $\overline{\mathcal{Y}}^t(\mathcal{X}^t)$ , which is a regular function by construction, makes the structure of the resulting formula more transparent and convenient for practical finding *singular*  $k$ .

The claim that  $\mathcal{X}^t \in \text{Rad}$  (the first part of (ii)) has interesting applications including the case of roots of unity. Note that the radical is *always* of finite codimension as  $q$  is a root of unity. However,  $\text{rad}(q, t) = \Pi_{\tilde{R}}$  becomes zero only for finitely many  $k$ . The “singular”  $k$  form finitely many sequences  $\{k_\nu + \mathbb{Z}_+\}$  such that applying the shift operator will eventually make  $\mathcal{X}^t \in \text{Rad}$ .  $\square$

## 12. IRREDUCIBILITY OF $\mathcal{V}$

We study the irreducibility of the polynomial representation  $\mathcal{V}$  under the assumption that the radical  $\text{Rad}$  of the form  $\{ , \}$  is zero. Recall that  $\tilde{R}^0, \tilde{R}^{-1}$  are from (9.7) and (10.27);  $\tilde{R}_+^0, \tilde{R}_+^{-1}$  and  $\tilde{R}_+^0[-], \tilde{R}_+^{-1}[-]$  denote positive roots in these sets and those with negative non-affine components.

The radical is of finite codimension when  $q$  is a root of unity, so we shall consider only generic  $q$ . From now on we assume that  $q = q_{\text{sht}}$  is not a root of unity, neither is  $q_{\text{lng}} = q^{\nu_{\text{lng}}}$ .

**12.1. Properties of  $\tilde{R}^0, \tilde{R}^{-1}$ .** We will need to develop some tools of combinatorial nature. Recall that the relation  $(\alpha_i, c + d) > 0$  is necessary and sufficient for  $s_i \pi_c$  to be represented in the form  $\pi_b$  for  $l(\pi_b) = l(\pi_c) + 1$ ; see Proposition 1.6.

$$\lambda(\pi_b) \setminus \lambda(\pi_c) = \pi_c^{-1}(\alpha_i) = \tilde{\alpha} + [0, (\alpha_i, c)] = [u_c(\alpha_i), \delta_{i0} + (\alpha_i, c)]$$

for the Kronecker delta, where  $\tilde{\alpha} \stackrel{\text{def}}{=} u_c(\alpha_i), (\alpha_i, c) = (\tilde{\alpha}, c_-)$ . The nonaffine component of  $\tilde{\alpha}$  must be negative since  $\tilde{\alpha}$  belongs to  $\lambda(\pi_b)$ .

For  $c_\# = \pi_c((- \rho_k)) = c - u_c^{-1}(\rho_k)$ ,

$$(\tilde{\alpha}, c_- - \rho_k + d) = (\alpha_i, c_\# + d) = (\tilde{\alpha} + [0, (c, \alpha_i)], -\rho_k + d),$$

where we transform:  $(\tilde{\alpha}, c_- + d) = (u_c(\alpha_i), u_c(c) + d) = (\alpha_i, c + d)$ .

The following lemma uses the notation from the tables from [B].

**Lemma 12.1.** *For  $c \in B$  and an index  $0 \leq i \leq n$  satisfying the inequality  $(\alpha_i, c + d) = (\alpha, c_-) + \delta_{i0} > 0$ , let*

$$(12.15) \quad t_\alpha^{-1} q^{(\tilde{\alpha}, c_- - \rho_k + d)} \stackrel{\text{def}}{=} q_\alpha^{(\tilde{\alpha}^\vee, c_- - \rho_k + d) - k_\alpha} = 1 = t_i^{-1} q^{(\alpha_i, c_\# + d)}$$

for  $\tilde{\alpha} = u_c(\alpha_i) = [\alpha, \delta_{i0}]$ ,  $\alpha = u_c(\alpha_i)$  for  $i > 0$  or  $\alpha = u_c(-\vartheta)$  for  $i = 0$ . Equivalently,  $\tilde{\alpha}' \stackrel{\text{def}}{=} \pi_c^{-1}(\alpha_i) = [\alpha, \delta_{i0} + (\alpha, c_-)] \in \tilde{R}^1$ ; see (10.22). Then  $-\alpha$  is positive and cannot be simple.

(i) If  $-\alpha$  is long then always  $\alpha + \alpha_l = (\nu_\alpha / \nu_\beta) \beta$  for proper  $-\beta \in R_+$  and long  $\alpha_l$  ( $l > 0$ ), where  $\beta$  can be taken long unless for  $\tilde{R} = \tilde{C}_n$  (and for long  $-\alpha$ ). In the non-simply-laced case and for short  $-\alpha$ , we assume that such representation exists with short  $\beta, \alpha_l$ . Let  $\tilde{\alpha}' = [\alpha, \nu_\alpha j']$ ,  $\tilde{\beta}' \stackrel{\text{def}}{=} [\beta, \nu_\beta j']$  for  $\beta = (\nu_\beta / \nu_\alpha)(\alpha + \alpha_l)$ . Then

$$(12.16) \quad q_\beta^{\frac{\nu_\alpha}{\nu_\beta}((\tilde{\beta}')^\vee, -\rho_k + d)} = 1, \quad \text{i.e., } \tilde{\beta}' \in \mathbb{Q}\langle \tilde{R}^0 \rangle$$

for the  $\mathbb{Q}$ -span  $\mathbb{Q}\langle \tilde{R}^0 \rangle$  of  $\tilde{R}^0$ . Here  $\tilde{\beta}' \in \tilde{R}^0$  unless  $\nu_\alpha / \nu_\beta > 1$  (see Lemma 12.2, (i) below).

If  $\tilde{\beta}' \in \tilde{R}^0$ , then  $\tilde{\beta}' \notin \lambda(\pi_c)$ . Moreover, one can find  $b \in B$  and the index  $i' \geq 0$  such that  $l(\pi_b) = l(\pi_b \pi_c^{-1}) + l(\pi_c)$  and

$$(12.17) \quad q^{(\alpha_{i'}, b_\# + d)} = 1, \quad (\alpha_{i'}, b + d) > 0, \quad (u_b(\alpha_{i'}), u_c(\alpha_i)) > 0, \quad \text{i.e.,}$$

$$\lambda(\pi_b) \not\supset \tilde{\beta}' = \pi_b^{-1}(\alpha_{i'}) \in \lambda(s_{i'} \pi_b) = \{\tilde{\beta}', \lambda(\pi_b)\},$$

where  $\beta = u_b(\alpha_{i'})$  for  $i' > 0$  or  $\beta = u_b(-\vartheta)$  for  $i' = 0$ .

(ii) Provided (11.14), the assumption from (i) holds for any long  $-\alpha$  and for short  $-\alpha \in R_+$  such that  $(\alpha, \rho_k) + k_{\text{sht}} \in 1 + \mathbb{Z}_+$  unless  $\alpha$  is from the following list:

$$-\alpha = \varepsilon_{n-g}, \quad g = 1, \dots, n-1, \quad (\alpha, \rho_k) + k_{\text{sht}} = -2gk_{\text{lng}} \in 1 + \mathbb{Z}_+, \quad (\tilde{B}_n)$$

$$-\alpha = \epsilon_{n-1} + \epsilon_n, \quad (\alpha, \rho_k) + k_{\text{sht}} = -2k_{\text{lng}} \in 1 + \mathbb{Z}_+, \quad (\tilde{C}_n)$$

$$-\alpha = \alpha_1 + \alpha_2, \quad (\alpha, \rho_k) + k_{\text{sht}} = -3k_{\text{lng}} \in 1 + \mathbb{Z}_+, \quad (\tilde{G}_2)$$

$$-\alpha = \alpha_3 + \alpha_2 \text{ or } -\alpha = \alpha_3 + \alpha_2 + \alpha_1 \quad (\text{see [B]}), \quad (\tilde{F}_4)$$

$$\text{where } (\alpha, \rho_k) + k_{\text{sht}} = -2k_{\text{lng}} \text{ or } -4k_{\text{lng}} \in 1 + \mathbb{Z}_+,$$

$$-\alpha = 1221, \quad (\alpha, \rho_k) + k_{\text{sht}} = -6k_{\text{lng}} - 2k_{\text{sht}} \in 1 + \mathbb{Z}_+. \quad (\tilde{F}_4)$$

In the case of long  $\alpha$  such that  $\nu_\alpha / \nu_\beta > 1$ , (11.14) results in  $\tilde{\beta}' \in \tilde{R}^0$ .



(iii) Among the cases listed in (ii) subject to (11.14), the radical can be zero only when  $k_{\text{lng}} = -j \in -1 - \mathbb{Z}_+$ , which implies that  $[-\alpha_p, \nu_\alpha j] \in \tilde{R}_+^0$  for any long simple root  $\alpha_p$  ( $p > 0$ ), or in the following subcases:

$$(12.18) \quad \tilde{B}_n : \quad k_{\text{lng}} = -\frac{s}{2g} \quad \text{for } n > g > \frac{n}{2}, s \in \mathbb{N}, (s, 2g) = 1,$$

$$\tilde{G}_2 : \quad k_{\text{lng}} = -s/3, \quad \text{where } s \in 1 + \mathbb{Z}_+, (s, 3) = 1,$$

$$\tilde{F}_4 : \quad k_{\text{lng}} = -s/4, \quad \text{where } s \in 1 + \mathbb{Z}_+, (s, 2) = 1,$$

$$\tilde{F}_4 : \quad 3k_{\text{lng}} + k_{\text{sht}} = -s/2 \quad \text{for } s \in 1 + \mathbb{Z}_+, (s, 2) = 1.$$

In the case of  $\tilde{F}_4$ , it is also possible that  $3k_{\text{lng}} + k_{\text{sht}} = -j \in -1 - \mathbb{Z}_+$  (then  $\text{Rad} = \{0\}$  for generic  $k_{\text{sht}}$ ), but in this case the long root  $[-1220, 2j]$  belongs to  $\tilde{R}_+^0$ .

*Proof.* Generally,  $t_i^{\pm 1} q^{(\alpha_i, c_{\sharp} + d)} = 1$  is equivalent to

$$(12.19) \quad \begin{aligned} q_{\alpha}^{\pm k_{\alpha} + (\alpha^{\vee}, c_- - \rho_k)} &= 1 \quad \text{for } \alpha = u_c(\alpha_i) \quad (i > 0), \\ q_{\alpha}^{\pm k_{\alpha} + (\alpha, c_- - \rho_k) + 1} &= 1 \quad \text{for } \alpha = u_c(-\vartheta) \quad (i = 0). \end{aligned}$$

We use the definition of  $c_{\sharp}$  and the relation  $u_c(c) = c_- \in B_-$ .

Recall that  $\alpha < 0$  in (12.19) and (12.15) due to

$$(12.20) \quad (\alpha_i, c + d) > 0 \Rightarrow (u_c(\alpha_i), c_- + d) > 0,$$

which is obvious as  $i > 0$ . However if  $i = 0$  then it may happen that  $(u_c(\alpha_i), c_-) = 0 = (c, \vartheta)$ . In the latter case the sign of  $\alpha$  cannot be determined from (12.20) and we need to use:

$$(c, \vartheta) = 0 \Rightarrow u_c(\vartheta) > 0, \quad \text{since } \lambda(u_c) = \{\alpha \in R_+, (c, \alpha) > 0\}.$$

The root  $-\alpha$  cannot be a simple root. Indeed, otherwise  $\alpha = -\alpha_p$  for  $p > 0$  and

$$1 = q_{\alpha}^{-k_{\alpha} + (\alpha^{\vee}, c_- - \rho_k) + \delta_{i0}} = q_{\alpha}^{(-\alpha_p^{\vee}, c_-) + \delta_{i0}} \Rightarrow (\alpha, c_-) + \delta_{i0} = 0,$$

which contradicts the assumption from (i); here we use that  $q_{\alpha}$  is not a root of unity. By the way, this argument shows that the set from (i) is empty for  $\tilde{A}_1$ .

Now, imposing (i), let  $\frac{\nu_{\alpha}}{\nu_{\beta}}\beta = \alpha + \alpha_l$  for certain  $l > 0$  such that  $\nu_{\alpha_l} = \nu_{\alpha}$ , i.e.,  $\alpha$  and  $\alpha_l$  have to be of the same length ( $\nu_{\beta} = \nu_{\alpha}$  unless for long  $\alpha$  in  $\tilde{C}_n$ ). This representation guaranties that  $(\beta, \alpha) > 0$  and  $s_{\alpha}(\beta) = \alpha_l > 0$  that directly leads to the inequality  $(u_b(\alpha_{i'}), u_c(\alpha_i)) > 0$  stated in (12.17).

Let  $\tilde{\beta}' = \frac{\nu_\beta}{\nu_\alpha}(\tilde{\alpha}' + \alpha_l) = [\beta, \nu_\beta j']$  for  $\tilde{\alpha} = [\alpha, \nu_\alpha j']$ . Then

$$\begin{aligned} q_\beta^{\frac{\nu_\alpha}{\nu_\beta}((\tilde{\beta}')^\vee, -\rho_k + d)} &= q^{(\frac{\nu_\alpha}{\nu_\beta}\beta, -\rho_k) + \nu_\alpha j'} \\ &= q_\alpha^{-k_\alpha + ((\frac{\nu_\alpha}{\nu_\beta}\beta - \alpha_l)^\vee, -\rho_k) + j'} = q_\alpha^{-k_\alpha + (\alpha^\vee, -\rho_k) + j'} = 1. \end{aligned}$$

Unless  $\nu_\alpha/\nu_\beta = 2$  for  $\tilde{C}_n$ , we conclude that  $\tilde{\beta}' \in \tilde{R}^0$ . If (11.14) is assumed, then it holds always (we can omit the factor  $\nu_\alpha/\nu_\beta$ ). Recall that  $q^{(\tilde{\alpha}, \cdot)} = q_\alpha^{(\tilde{\alpha}^\vee, \cdot)}$  by definition and all such powers must be expressed as products of powers of  $q$  and  $t_\nu$ .

Let us check that  $\tilde{\beta}' \notin \lambda(\pi_c)$ . It can be seen directly from (1.25), but it is simpler to use that

$$\frac{\nu_\alpha}{\nu_\beta} \tilde{\beta}' = \frac{\nu_\alpha}{\nu_\beta} [\beta, \nu_\beta j'] = \tilde{\alpha}' + \alpha_l$$

and apply Theorem 2.1, (b'). The root  $\tilde{\beta}'$  can appear in  $\lambda(s_i \pi_c) = \{\tilde{\alpha}', \lambda(\pi_c)\}$  only *after*  $\tilde{\alpha}'$ , which is the last in this set, since  $\alpha_l$  does not belong to  $\lambda(\pi_c)$  (and to any  $\lambda(\pi_a)$ ).

To construct  $b$ , it suffices to use an *arbitrary*  $a \in B$  satisfying  $\pi_a = \hat{w}s_i \pi_c$  such that  $l(\pi_a) = l(\hat{w}) + 1 + l(\pi_c)$  and  $\tilde{\beta}' \in \lambda(\pi_a)$ ; we simply make  $(a_-, \beta)$  sufficiently large. Then for any reduced decomposition  $\pi_a = \pi_r s_{i_m} \cdots s_{i_1}$ , the partial products  $s_{i_h} \cdots s_{i_2} s_{i_1}$  remain in  $\pi_B$  for any  $h \leq m$ . Note that these products also remain in  $\pi_b$  if we multiply them by *any*  $\pi_{r'}$  on the left.

We can assume that a reduced decomposition for  $\pi_a$  extends that for  $\pi_c$ . Let  $s_{i'} \pi_b$  be the first partial product “after”  $\pi_c$  such that  $\tilde{\beta}' \in \lambda(s_{i'} \pi_b)$ . It gives the required  $b$  since the affine root  $\tilde{\beta}'$  can appear in the  $\lambda$ -sets of the partial products only after  $\lambda(\pi_c)$ .

Claim (ii) is checked using the tables from [B]. Concerning (iii), we need to examine the special short  $\alpha$  listed in (ii) and employ Theorem 11.8.  $\square$

**12.2. A generalization (any  $t$ ).** It is not too difficult to extend the lemma to a general setting when  $q$  is assumed to be generic (not a root of unity) but condition (11.14) is omitted in (ii), (iii). The main change we need is at the end of (ii), where it is stated that in the case of long  $\alpha$  such that  $\nu_\alpha/\nu_\beta > 1$  condition (11.14) results in  $\tilde{\beta}' \in \tilde{R}^0$ . Now we

cannot use it. First of all, we need to rewrite the formulas in the lemma by adding  $q$ .

The  $\mathbb{Z}$ -integrality conditions in (ii) become:

$$(12.21) \quad \begin{aligned} t_{\text{lng}}^{-g} &\in q^{1+\mathbb{Z}_+} \text{ for } \tilde{B}_n, \quad t_{\text{lng}}^{-1} \in q^{1+\mathbb{Z}_+} \text{ for } \tilde{C}_n, \\ t_{\text{lng}}^{-1} &\in q^{1+\mathbb{Z}_+} \text{ in the case of } \tilde{G}_2, \text{ and for } \tilde{F}_4 : \\ t_{\text{lng}}^{-1} &\in q^{1+\mathbb{Z}_+}, \quad t_{\text{lng}}^{-2} \in q^{1+\mathbb{Z}_+}, \quad t_{\text{lng}}^{-3} t_{\text{sht}}^{-2} \in q^{1+\mathbb{Z}_+}. \end{aligned}$$

Recall that  $t_{\text{lng}} = q_{\text{lng}}^{k_{\text{lng}}}$  for  $q_{\text{lng}} = q^{\nu_{\text{lng}}}$ . The condition  $k_{\text{lng}} \in -1 - \mathbb{Z}_+$  now reads as  $t_{\text{lng}}^{-1} \in q^{2+2\mathbb{Z}_+}$ .

Similarly, the list from (iii) becomes:

$$(12.22) \quad \begin{aligned} \tilde{B}_n : \quad &t_{\text{lng}}^{-g} \in q^{1+\mathbb{Z}_+} \setminus q^{2+2\mathbb{Z}_+}, \quad \tilde{G}_2 : \quad t_{\text{lng}}^{-1} \in q^{1+\mathbb{Z}_+} \setminus q^{3+3\mathbb{Z}_+}, \\ \tilde{F}_4 : \quad &t_{\text{lng}}^{-2} \in q^{1+\mathbb{Z}_+} \setminus q^{2+2\mathbb{Z}_+} \text{ or } t_{\text{lng}}^{-3} t_{\text{sht}}^{-2} \in q^{1+\mathbb{Z}_+} \setminus q^{2+2\mathbb{Z}_+}, \end{aligned}$$

where  $n/2 < g < n$  and  $t_{\text{lng}}^{-g'} \notin q^{1+\mathbb{Z}_+}$  with any  $\mathbb{Z}_+ \ni g' < g$  for  $\tilde{B}_n$ .

Let us describe the cases when  $\alpha$  is *long* and  $\nu_\alpha/\nu_\beta > 1$  in (i); they are exactly those dual to the cases listed in (ii). We continue to assume that  $q$  is not a root of unity but do not impose the  $q, t$ -generality condition (11.14). Then (12.15) reads as

$$\begin{aligned} q^{\nu_{\text{lng}}((\alpha^\vee, \rho_k) + k_{\text{lng}})} &\in q^{-\nu_{\text{lng}}(1+\mathbb{Z}_+)}, \\ 1 &\in q^{(\alpha^\vee, \rho_k) + k_{\text{lng}} + 1 + \mathbb{Z}_+} \Rightarrow \tilde{\beta}' \in \tilde{R}^0. \end{aligned}$$

Recall that the first condition becomes  $(\alpha^\vee, \rho_k) + k_{\text{lng}} \in -1 - \mathbb{Z}_+$  and always implies  $\tilde{\beta}' \in \tilde{R}^0$  if (11.14) is imposed.

**Lemma 12.2.** (i) *The cases  $\nu_\alpha/\nu_\beta > 1$  are as follows:*

$$\begin{aligned} -\alpha = 2\varepsilon_{n-g}, \quad 1 \leq g \leq n-1, \quad (\alpha^\vee, \rho_k) + k_{\text{lng}} &= -gk_{\text{sht}}, & (\tilde{C}_n) \\ \text{where condition (12.15) becomes } t_{\text{sht}}^{-2g} &\in q^{2+2\mathbb{Z}_+}, \\ -\alpha = \epsilon_{n-1} + \epsilon_n, \quad (\alpha^\vee, \rho_k) + k_{\text{lng}} &= -k_{\text{sht}}, \quad t_{\text{sht}}^{-2} \in q^{2+2\mathbb{Z}_+}, & (\tilde{B}_n) \\ -\alpha = 3\alpha_1 + \alpha_2, \quad (\alpha^\vee, \rho_k) + k_{\text{lng}} &= -k_{\text{sht}}, \quad t_{\text{sht}}^{-3} \in q^{3+3\mathbb{Z}_+}, & (\tilde{G}_2) \\ -\alpha = \alpha_2 + 2\alpha_3 \text{ or } -\alpha = \alpha_2 + 2\alpha_3 + 2\alpha_4, &\text{ respectively,} & (\tilde{F}_4) \\ \text{for } (\alpha^\vee, \rho_k) + k_{\text{lng}} = -gk_{\text{sht}}, \quad t_{\text{sht}}^{-2g} &\in q^{2+2\mathbb{Z}_+} \text{ as } g = 1, 2, \\ -\alpha = 1242, \quad (\alpha^\vee, \rho_k) + k_{\text{lng}} &= -2k_{\text{lng}} - 3k_{\text{sht}}, \quad t_{\text{lng}}^{-2} t_{\text{sht}}^{-6} \in q^{2+2\mathbb{Z}_+}. & (\tilde{F}_4). \end{aligned}$$

Let  $\tau = \tau(-\alpha) \stackrel{\text{def}}{=} t_{\text{sht}}$  in all these cases but the last, where  $\tau(1242) \stackrel{\text{def}}{=} t_{\text{lng}} t_{\text{sht}}^3$ . We set  $g = 1$  unless stated otherwise (namely, when  $\tilde{R} = \tilde{C}_n$  and for  $\tilde{F}_4$  as  $g = 2$ ). In these notations, if  $\tilde{\beta}' \notin \tilde{R}_+^0$  (see above) then  $\tau^{-g\nu} \in q^{\nu+\nu\mathbb{Z}_+}$  for  $\nu = \nu_{\text{lng}}$ .

(ii) Continuing (i), let  $\text{Rad} = \{0\}$ . Then either  $t_{\text{sht}} \in q^{-1-\mathbb{Z}_+}$  or one of the following conditions hold :

$$\begin{aligned}
 (12.23) \quad \tilde{C}_n : \quad & t_{\text{sht}}^g \in -q^{-1-\mathbb{Z}_+}, \quad \text{where } n > g > \frac{n}{2} \\
 & \text{and } t_{\text{sht}}^{g'} \notin \pm q^{-1-\mathbb{Z}_+} \quad \text{for } \mathbb{Z}_+ \ni g' < g, \\
 \tilde{G}_2 : \quad & t_{\text{sht}} \in \rho q^{-1-\mathbb{Z}_+} \quad \text{for } \rho^2 + \rho + 1 = 0, \\
 \tilde{F}_4 : \quad & t_{\text{sht}}^2 \in -q^{-1-\mathbb{Z}_+} \quad \text{for } -\alpha = 0122, \\
 \tilde{F}_4 : \quad & t_{\text{lng}} t_{\text{sht}}^3 \in -q^{-1-\mathbb{Z}_+} \quad \text{as } -\alpha = 1242.
 \end{aligned}$$

In the case  $-\alpha = \alpha_2 + 2\alpha_3$  for  $\tilde{F}_4$  from (i),  $t_{\text{lng}} t_{\text{sht}}^3 \in q^{-1-\mathbb{Z}_+}$  does not result in  $\text{Rad} \neq \{0\}$ ; however, if it holds, then  $[-0121, j] \in \tilde{R}_+^0$  for proper  $j \in 1 + \mathbb{Z}_+$ .

□

**12.3. Zigzag paths.** Following the last section, we add more detail concerning the combinatorial structure of  $\tilde{R}_+^0$  and  $\tilde{R}_+^{-1}$ , the sets of positive roots in  $\tilde{R}^0$  and  $\tilde{R}^{-1}$ . We need to make more transparent the procedure from Lemma 12.1 and also provide the combinatorial tools for Theorem 12.7 below.

The condition that  $\tilde{\alpha}' \in \tilde{R}_+^{-1}$  corresponds to  $\tilde{\beta}' \in \tilde{R}_+^0$  (in the sense of (i)) has the following general meaning. Always,  $s_{\tilde{\beta}'}(\tilde{\alpha}')$  belongs to  $\tilde{R}^1$  (it holds for any  $\tilde{w} \in \tilde{W}^0$ ). Then  $\tilde{\beta}' = [\beta', \nu_{\beta'} j]$  in (i) are precisely those satisfying:

$$\begin{aligned}
 (12.24) \quad (a) : \quad & s_{\tilde{\beta}'}(\tilde{\alpha}') = -\alpha_l \quad (l > 0), \quad -\beta' \in R_+, \quad j > 0, \\
 (b) : \quad & \tilde{\alpha}' = \text{short} \Rightarrow \tilde{\beta}' = \text{short}.
 \end{aligned}$$

To avoid misunderstanding, we use here the notations  $\beta'$  and  $\alpha'$  for the non-affine components of  $\tilde{\beta}'$  and  $\tilde{\alpha}'$  instead of  $\beta$  and  $\alpha$  used in (i).

The set of all  $\alpha_l$  satisfying (12.24) for some  $\tilde{\alpha}' = [\alpha', \nu_\alpha j] \in \tilde{R}_+^{-1}[-]$ , i.e., with *negative* non-affine components  $\alpha'$ , is described as follows:

$$(a') : (\alpha_l, \tilde{\beta}') = (\alpha_l, \beta') > 0, \quad (b') : \beta' = \text{long} \Rightarrow \alpha_l = \text{long}.$$

Always  $s_l(\tilde{\beta}') = \tilde{\beta}' - \alpha_l$  due to (a, b), but  $s_l(\tilde{\beta}')$  may be different from  $\tilde{\alpha}' = -s_{\tilde{\beta}'}(\alpha_l)$  in the non-simply-laced case.

If (a', b') hold, then we connect  $\tilde{\alpha}'$  and  $\tilde{\beta}'$  by a *link*. We say that different  $\tilde{\beta}', \tilde{\beta}'' \in \tilde{R}_+^0[-]$  are  $\tilde{R}^0$ -*neighbors* if they are connected with the same  $\tilde{\alpha}' \in \tilde{R}_+^{-1}[-]$  by *links*; similarly,  $\tilde{\alpha}', \tilde{\alpha}'' \in \tilde{R}_+^{-1}[-]$  are called  $\tilde{R}^{-1}$ -*neighbors* if they correspond to different  $\alpha_l$  and  $\alpha_m$  and the same  $\tilde{\beta}'$ . By a **zigzag**, we mean a *connected* subgraph of the graph of all roots from  $\tilde{R}^{-1}, \tilde{R}^0$  taken as the vertices with the *links* considered as the edges; generally, it may be a tree or a loop.

Note that if  $\tilde{\alpha}', \tilde{\alpha}''$  are *neighbors* in  $\tilde{R}_+^{-1}[-]$ , then

$$(12.25) \quad \{(\alpha_l, -\beta') < 0 > (\alpha_m, -\beta')\}, \text{ which implies } (\alpha_l, \alpha_m) = 0;$$

recall that  $-\beta' \in R_+$ . Indeed, if  $\alpha_l$  and  $\alpha_m$  are connected in the Dynkin (non-affine) diagram  $\Gamma$  and  $\nu_l \geq \nu_m$ , then  $s_m(\alpha_l) = \alpha_l + (\nu_l/\nu_m)\alpha_m$  belongs to  $R_+$  and

$$s_{-\beta'}(\alpha_l + (\nu_l/\nu_m)\alpha_m) = \alpha_l + (\nu_l/\nu_m)\alpha_m + M(-\beta'),$$

where  $M = 2(\nu_l/\nu_m)$  as  $\nu_{\beta'} = \nu_m$  and  $M = 1 + (\nu_l/\nu_m)$  as  $\nu_{\beta'} \geq \nu_l$ . However, when  $-\beta'$  is long or when all three roots are of the same length, then  $M < 2$ , which contradicts to the above formulas for  $M$ .

For instance, the orthogonality from (12.25) gives that the number of *links* from a given  $\tilde{\beta}' \in \tilde{R}_+^0[-]$  cannot be greater than the maximum degree of the vertices in  $\Gamma$  considered as a tree (ignoring the multiplicities of laces), which is 2 unless for  $D, E$ .

Actually, the positivity  $-\beta' > 0$  was used above only in the last inequality  $M < 2$ . For negative  $-\beta'$  satisfying the inequalities from (12.25),  $M$  can be 2 for neighboring  $\alpha_l, \alpha_m$  if  $-\beta' = -\alpha_l - \alpha_m$  and  $\alpha_l, \alpha_m$  are of the same length. We obtain an inversion of (12.25):

$$(12.26) \quad \begin{aligned} &(\alpha, \alpha_l) > 0 < (\alpha, \alpha_m) \text{ for } \alpha \in R_+ \Rightarrow \\ &(\alpha_l, \alpha_m) = 0 \text{ unless } \alpha = \alpha_l + \alpha_m \text{ and } \nu_l = \nu_m. \end{aligned}$$

See (12.29) below. As a matter of fact, (12.25) and (12.26) are standard facts on subsystems of affine root system; the latter can be deduced from the former by setting  $-\beta' = [-\alpha, \nu_\alpha]$ .

**Restricted zigzags.** We will assume that  $t_\nu \notin q^{-\nu(1+\mathbb{Z}_+)}$  in the next lemma for any  $\nu = \nu_{\text{sht}}, \nu_{\text{lng}}$ ; the cases  $[-1220, 2j] \in \tilde{R}^0$  and  $[-0121, j] \in \tilde{R}^0$  for  $\tilde{F}_4$  will be omitted. Let us also exclude the cases in (12.18) or, more generally, in (12.22) and (12.23). We can now make (b) more restrictive:

$$(12.27) \quad (b'') : \nu_{\beta'} = \nu_{\alpha'} = \nu_l \text{ and } \tilde{\alpha}' = s_l(\tilde{\beta}') = \tilde{\beta}' - \alpha_l.$$

Respectively, we will use *restricted* links and zigzags. Note that one can find  $\alpha_l$  to “go” from *any*  $\tilde{\alpha}'$  to  $\tilde{\beta}'$  under  $(b'')$  since we excluded the cases where it does not hold. Also,  $((\tilde{\beta}')^\vee, \rho) = ((\tilde{\beta}'')^\vee, \rho)$  if  $\tilde{\beta}'$  and  $\tilde{\beta}''$  are  $\tilde{R}^0$ -neighbors and

$$(12.28) \quad \tilde{\beta}'' = \tilde{\beta}' + \alpha_{l'} - \alpha_l, \quad \beta'' = \beta' - \alpha_{l'} + \alpha_{l'}, \quad \nu_l = \nu_{l'}$$

due to (12.25), (12.27), where  $\tilde{\beta}' - \alpha_l = \tilde{\alpha}' = \tilde{\beta}'' - \alpha_{l'}$ . Using (12.26):

$$(12.29) \quad (\alpha_l, \alpha_{l'}) = 0 \text{ if } -\alpha' \neq \alpha_l + \alpha_{l'} \text{ for neighboring } \alpha_l, \alpha_{l'}.$$

We will not use this fact but it is helpful in understanding the combinatorial structure of  $\tilde{R}^{0,-1}$ .

We mention that these definitions resemble the BGG-resolutions. Namely,  $s_l s_m(\tilde{\beta}') = \tilde{\beta}' - \alpha_l - \alpha_m$  is *link*-connected with  $\tilde{\alpha}'$  and  $\tilde{\alpha}''$ , and  $s_l s_{l'}(\tilde{\alpha}') = \tilde{\alpha}' - \alpha_l - \alpha_{l'}$  is connected with  $\tilde{\beta}'$  and  $\tilde{\beta}''$  in the corresponding cases. This is an example of the BGG-type squares.

We see that the *zigzag-connectivity* is essentially a non-affine notion (if  $q$  is not a root of unity) with close relations to the classical theory, especially under (12.27). Given  $p \in 1 + \mathbb{Z}_+$ , let

$$\tilde{R}_+^{0,\pm 1}(p) \text{ be the set of all } \tilde{\gamma}' = [\gamma', p] \in \tilde{R}_+^{0,\pm 1}[-].$$

**Lemma 12.3.** *Provided (12.27), a restricted zigzag belongs to the set  $\tilde{R}^0(p) \cup \tilde{R}^{-1}(p)$  for proper  $p \in 1 + \mathbb{Z}_+$ . If it is maximal, i.e., not a part of a larger zigzag, then it contains a boundary root from  $\tilde{R}^0$  that has a unique link to  $\tilde{R}^{-1}$  by definition.*

*Proof.* If  $\nu_{\text{lng}} k_{\text{lng}} = k_{\text{sht}}$  (that includes the simply-laced case), the non-affine components of the roots from  $\tilde{R}_+^0(p)$  constitute a subset  $-R_{ht}$

of  $R_-$  of the roots  $\beta$  of fixed height  $ht \stackrel{\text{def}}{=} (-\beta', \rho^\vee)$ . Similarly, the set  $\tilde{R}_+^1(p)$  will be  $-R_{ht+1}$ . Note that the zigzags with  $\beta'$ -roots of different length do not intersect.

The claim of the lemma in this case becomes a simple statement about *zigzags* in  $-(R_{ht} \cup R_{ht+1})$ , which is straightforward. As a matter of fact, this statement is a combinatorial version of (7.13). Let us outline the deduction of (7.13) from the claim of the lemma considered for abstract *zigzags* in  $-(R_{ht} \cup R_{ht+1})$  for an arbitrary  $ht$ .

We take a maximal *segment of  $\tilde{R}^0$ -neighbors* in  $-R_{ht}$ , a longest chain of consecutive  $\tilde{R}^0$ -neighbors there. It contains all roots in  $-R_{ht}$  unless in the cases  $D, E$ , where we need to add an additional *segment* with the corresponding zigzag that begins at  $-R_{ht}$  and ends at  $-R_{ht}$  too. The zigzag corresponding to the maximal segment can be with one or two endpoints from  $-R_{ht}$ . Correspondingly,  $|R_{ht}| - |R_{ht+1}|$  is 0, 1, or 2 in the case of  $D_{\text{even}}$ . It gives the required structure of the right-hand side of (7.13) since the denominator is known. It is quite possible that this approach to proving (7.13) is known, but we found no proper references.

The sets  $\tilde{R}_+^0(p)$  become the greatest as  $\nu_{\text{lng}} k_{\text{lng}} = k_{\text{sht}}$  (it is the so-called case of equal labels). If the latter constraint is not imposed, then we inspect the cases of  $B, C$  directly and use that each  $\tilde{R}_+^0(p)$  can contain only one or two different roots for  $F, G$  (if not empty). Cf. (7.15).  $\square$

The claim of the lemma is likely to hold (essentially) without imposing (12.27). We expect that there must be a general reason for any maximal *zigzag* in  $\tilde{R}_+^0 \cup \tilde{R}_+^{-1}$  to contain the *boundary* points from  $\tilde{R}_+^0$ , which are the ones with only one link to  $\tilde{R}_+^{-1}$ .

The following corollary seems of general nature too, but we will state it assuming that the radical is zero and avoiding the exceptional cases from (12.18) or, more generally, from (12.22) and (12.23). It can somewhat simplify proving Theorem 12.7 when being used instead of Lemma 12.3, for instance, Lemma 1.2 and Corollary 3.6 can be avoided.

**Corollary 12.4.** *Provided (11.14), we assume that the radical is zero apart from the cases listed in (12.18). In the setting with  $q$ , the cases from (12.22) and (12.23) must be excluded. We also assume that  $t \neq q^{-1-\mathbb{Z}_+}$  in the simply-laced case.*

Then there exists  $\pi_a$  for  $a \in B$  and its reduced decomposition such that  $\lambda(\pi_a)$  contains the sets  $\tilde{R}_+^0, \tilde{R}_+^{\pm 1}$  and satisfies the following  $\tilde{R}^{-1} \rightarrow \tilde{R}^0$ -alternation condition. Given any  $\tilde{\alpha}' \in \tilde{R}_+^{-1}$ , the first  $\tilde{\beta}' \in \tilde{R}_+^0$  after  $\tilde{\alpha}'$  in the sequence  $\lambda(\pi_a)$  satisfies  $(\tilde{\alpha}', \tilde{\beta}') > 0$  and no  $\tilde{\gamma}' \in \tilde{R}_+^{\pm 1}$  can be found in  $\lambda(\pi_a)$  between  $\tilde{\alpha}'$  and  $\tilde{\beta}'$  such that  $(\tilde{\gamma}', \tilde{\beta}') \neq 0$ .

*Sketch of the proof.* Here adding the condition  $t \notin q^{-1-\mathbb{Z}_+}$  in the simply-laced case is important; respectively,  $k \notin -1 - \mathbb{Z}_+$  if (11.14) is assumed.

The simplest counterexample to the claim of this corollary without imposing this condition is  $\tilde{D}_4$  (this condition is not needed for  $\tilde{A}$ ).

**Comment.** We note that if  $R$  is simply-laced and  $k \notin -1 - \mathbb{Z}_+$ , then the condition  $Rad = \{0\}$  implies that  $R_+^0$  is a set of pairwise orthogonal roots unless in the following two cases:

$$E_8 \text{ where } k = -g/11, -g/13 \notin \mathbb{Z} \text{ as } g \in \mathbb{N}$$

(similarly, in the setting with  $q$ ). Actually the alternating sequences are not needed in the proof of Theorem 12.7 when  $\tilde{R}_+^0$  consists of pairwise orthogonal roots; one can use part (ii) of the Key Lemma below.  $\square$

Essentially the procedure is as follows. We take a maximal zigzag path that begins with a proper *boundary* root from  $\tilde{R}_+^0$  (with a unique link to  $\tilde{R}_+^{-1}$ ) and transpose all consecutive pairs  $\tilde{\beta}' \rightarrow \tilde{\alpha}'$  in this path. Then the resulting sequence can be made a part of some  $\lambda(\pi_a)$ -sequence.

*The case of  $\tilde{E}_8$ .* The following example demonstrates the procedure. Let  $\tilde{R} = \tilde{E}_8$  and  $k = -g/13$ , where  $g \in \mathbb{N}$  and  $(g, 13) = 1$ . Then

$$\tilde{R}^0 = \{[-\beta, g], ht_\beta = (\beta, \rho) = 13\} \cup \{[-\beta, 2g], ht_\beta = 26\},$$

$$\tilde{R}^{-1} = \{[-\alpha, g], ht_\alpha = 14\} \cup \{[-\alpha, 2g], ht_\alpha = 27\} \text{ and also } \tilde{R}^1 = \{[-\gamma, g], ht_\gamma = 12\} \cup \{[-\gamma, 2g], ht_\gamma = 25\}.$$

The maximal zigzag we need is (in the notation from [B]):

$$(12.30) \quad \begin{array}{ccccccccc} 1232111^0 - & -1232211 - & -1232210^0 - & -1233210 - & & & & & \\ & 2 & 2 & 2 & 2 & & & & \\ -1233210^0 - & -1233211 - & -1232211^0 - & -1232221 - & -1222221^0. & & & & \\ & 1 & 1 & 1 & 1 & 1 & & & \end{array}$$



The  $\beta$ -components of the roots from  $\tilde{R}^0$  are marked by  $^0$ , the links are  $--$ ; here the endpoints are *boundary roots* (that have unique link-connections with  $\tilde{R}^{-1}$ ). The remaining roots are of height 26, 27:

$$\begin{array}{rcccl} 2465321^0 & = & 1232111^0 & + & 1233210^0, \\ 3 & & 2 & & 1 \\ 2465421 & = & 1232211 & + & 1233210^0. \\ 3 & & 2 & & 1 \end{array}$$

The corresponding  $\alpha, \beta$ -parts of the roots from the required  $\lambda$ -sequence intersected with  $\tilde{R}^{-1,0}$  can be made:

$$(12.31) \quad \begin{array}{cccccc} \{ 1222221^0, & 1232211^0, & 1232221, & 1233210^0, & 1233211, & \\ & 1 & 1 & 1 & 1 & 1 \\ 2465321^0, & 2465421, & 1232210^0, & 1233210, & 1232111^0, & 1232211 \}. \\ 3 & 3 & 2 & 2 & 2 & 2 \end{array}$$

Recall that the ordering of the roots in  $\lambda$ -sets is from right to left.

Here the roots from  $\tilde{R}_+^1$  can be inserted between the roots from  $\tilde{R}_+^0$  and the next ones from  $\tilde{R}_+^{-1}$ . Recall that  $|\tilde{R}_+^1| = |\tilde{R}_+^0|$  because of the assumption  $Rad = \{0\}$ . As a matter of fact, only one root from the 5 ones of height 12 in  $\tilde{R}_+^1$  can appear before the last two roots in (12.31).

Use formula (1.25) or Theorem 2.1,(ii) to justify that the resulting *sequence* can be made the intersection  $\lambda(\pi_a) \cap \{\tilde{R}_+^{-1} \cup \tilde{R}_+^0 \cup \tilde{R}_+^1\}$  for certain  $a \in B$ .

Using Proposition 1.4, one can construct geometrically some  $\pi_a$  satisfying the alternation condition. Let  $b_0 \in \mathfrak{C}^a, b_1 \in \mathfrak{C}$  for the *basic* affine and nonaffine Weyl chambers. Then  $L \stackrel{\text{def}}{=} \{tb_1 + (1-t)b_0, 0 \leq t \leq 1\} \subset \mathfrak{C}$ . Let  $[-\alpha, j] \in \tilde{R}_+^{-1,0,1}$ . Here  $j = g, 2g$ . We assume that  $(b_1, \alpha) > j$  for all such roots; automatically,  $(b_0, \alpha) < 1 \leq j$ . Then the ordering of the points

$$(12.32) \quad t = t(\alpha) = (j - (b_0, \alpha)) / (b_1 - b_0, \alpha), \quad \text{where } 0 < t < 1,$$

gives the ordering of the corresponding roots  $[-\alpha, j]$  in the  $\lambda$ -sequence associated with the segment  $L$ , which is a collection of the positive (all possible) affine roots such that their hyperplanes intersect  $L$ . The

condition  $b_1 \in \mathfrak{C}$  is necessary and sufficient to make the total set in the form  $\lambda(\pi_a)$  for certain  $a \in Q$ .

For instance, setting  $\epsilon = 0.1/g$ , the vectors  $b_0 = \epsilon\omega_5$  and  $b_1 = \epsilon(\omega_4 + \omega_5 + \omega_6 + \omega_7) + 2\epsilon\omega_8 + \omega_2$  (the notation is from [B]) result in the following  $\tilde{R}^{-1} \rightarrow \tilde{R}^0$ -alternating sequence:

$$(12.33) \quad \begin{array}{cccccc} \{ & 1232210^1, & 1232111^1, & 1222211^1, & 1233210^0, & 1232211^0, \\ & 1 & 1 & 1 & 1 & 1 \\ 1233211, & 1122221^1, & 1222221^0, & 1232221, & 2464321^1, & 2465321^0, \\ 1 & 1 & 1 & 1 & 3 & 3 \\ 2465421, & 1232110^1, & 1232210^0, & 1233210, & 1232111^0, & 1232211 \}, \\ 3 & 2 & 2 & 2 & 2 & 2 \end{array}$$

where the negatives of the nonaffine components of the roots from  $\tilde{R}_+^1$  are marked by <sup>1</sup>.

We note a relation of the alternating sequences to the so-called *non-crossing partitions*. The Coxeter element and its powers are useful for constructing required  $\pi_a$  and their reduced decompositions, namely, the formula  $w_0 = c^{h/2}$  from [B] for the Coxeter element  $c = s_n \cdots s_2 s_1$  and the Coxeter number  $h = 1 + (\rho, \vartheta)$  if  $R$  is not of type  $A$  and

$$w_0 = s_1(s_2 s_1) \cdots (s_{n-1} \cdots s_1)(s_n \cdots s_1) \text{ for } A_n.$$

□

**12.4. Key Lemma.** We suppose till the end of this section that  $t_{\text{lng}} \neq \pm 1 \neq t_{\text{sht}}$ . Here the condition  $\neq -1$  makes the sets  $\tilde{R}^{\pm 1}$  non-intersecting; the condition  $\neq 1$  is necessary in the next Key Lemma.

For  $\tilde{\alpha}' \in R_+^{-1}[-]$  from (12.15), we determine  $\bar{c}$  from  $\pi_{\bar{c}} = s_i \pi_c$  and suppose that there exists  $\tilde{\beta}' \in \tilde{R}_+^0$  (see there) such that  $\tilde{\beta}' \notin \lambda(\pi_c)$ ,  $(\tilde{\alpha}', \tilde{\beta}') > 0$  and, moreover,  $-s_{\tilde{\beta}'}(\tilde{\alpha}') > 0$ . We take the pair  $b, i'$  satisfying (12.17) assuming that

$$(12.34) \quad \pi_{\bar{b}} = s_{i'} \pi_b, \quad \lambda(\pi_b) \not\ni \tilde{\beta}' \in \lambda(\pi_{\bar{b}}) \text{ and}$$

there is no  $\tilde{\gamma}' \in R_+^{\pm 1}[-] \cup R_+^0[-]$  between  $\tilde{\alpha}'$  and  $\tilde{\beta}'$  in  $\lambda(\pi_{\bar{b}})$ .

Recall the notations:  $\pi_b = \hat{w} s_i \pi_c$  as  $l(\hat{w}) + l(\pi_c) = l(\pi_b)$ . We set  $\pi_d = \hat{w}(i') s_i \pi_c$  for  $\hat{w}(i') \stackrel{\text{def}}{=} \hat{w}^{-1} s_{i'} \hat{w}$ , i.e.,  $\pi_d$  is obtained when the inverse of the reduced decomposition of  $\hat{w}$  is added to that of  $s_{i'} \pi_b$ . One

can assume that the decomposition  $\widehat{w}^{-1}s_{i'}\widehat{w}$  is reduced (otherwise  $\pi_b$  can be diminished), in particular,  $(\widetilde{\alpha}', \widetilde{\gamma}') \neq 0$  for all  $\widetilde{\gamma}'$  between  $\widetilde{\alpha}'$  and  $\widetilde{\beta}'$  in the sequence  $\pi_b$ . However we do not suppose that  $(\widehat{w}^{-1}s_{i'}\widehat{w})s_i\pi_c$  is reduced. We set  $\widetilde{E}_d^\dagger = \Psi_{\widehat{w}(i')} \Psi_{s_i} E_c$ .

**Key Lemma 12.5.** (i) Let  $t_i \neq 1$  and  $\dim \mathcal{V}_c = 1$ . In the notation above, one of the following holds:

(a):  $-s_{\widetilde{\beta}'}(\widetilde{\alpha}') \in \widetilde{R}_+^1[-]$ , (b):  $-s_{\widetilde{\beta}'}(\widetilde{\alpha}') = \alpha_l$  ( $l > 0$ ), (c):  $-s_{\widetilde{\beta}'}(\widetilde{\alpha}') < 0$ . In either case,  $E' = \Psi_i \widetilde{E}_d^\dagger$  is a nonzero  $Y$ -eigenvector. If (b) holds, then  $E'$  is proportional to  $\widetilde{E}_c$ . Otherwise,  $E' \in \mathcal{V}_{\widetilde{d}}$  for  $\pi_{\widetilde{d}} = s_i\pi_d$ ; moreover,  $\dim \mathcal{V}_{\widetilde{d}} = 1$  and  $\widetilde{E}_{\widetilde{d}} = E_{\widetilde{d}}$  in case (a).

(ii) Generalizing, let us allow several pairs  $\{\widetilde{\beta}^v, \widetilde{\alpha}^v\}$ ,  $v = 1, \dots, p$  in (12.34) between  $\widetilde{\alpha}'$  and  $\widetilde{\beta}'$  connected in the same way as  $\widetilde{\alpha}'$  and  $\widetilde{\beta}'$  (Lemma 12.1). We also assume that  $(\widetilde{\beta}^v, \widetilde{\beta}^{v'}) = 0$  when  $v \neq v'$ . The polynomial  $E'$  is constructed as in (i), but now the singular intertwiners must be dropped when we go back from  $\widetilde{\beta}'$ . Then  $E'$  is a nonzero  $Y$ -eigenvector of the same weight as  $E_c$ .

*Proof.* Here  $\widetilde{E}_d^\dagger$  is not a  $Y_{\alpha_i}$ -eigenvector in  $\mathcal{V}_d$  due to  $(\widetilde{\alpha}', \widetilde{\beta}') \neq 0$ . In the transformation to  $E \mapsto E'$ , we multiply  $\widetilde{E}_d^\dagger$  by  $(\tau_+(T_i) - t_i^{1/2})$ ; then we use Corollary 10.8,(ii). The key claim is that  $\Psi_i \widetilde{E}_c = \text{const} \widetilde{E}_c$  for a nonzero constant in case (b); it holds because  $(\tau_+(T_i) - t_i^{1/2})\widetilde{\Psi}_{\pi_d}^\dagger(\mathbb{Q}_{q,t}1)$  belongs to  $\mathcal{V}_c$  and therefore must coincide with this space due to  $\dim \mathcal{V}_c = 1$ . In the case of (a) (we will omit (c)):

$$\widetilde{\Psi}_{\pi_{\widetilde{d}}} = \widetilde{\Psi}_i \widetilde{\Psi}_{\pi_d}^\dagger, \quad \dim \mathcal{V}_c = 1 \Rightarrow (Y_{\alpha_i} - t_i)\widetilde{\Psi}_i \widetilde{\Psi}_{\pi_d}^\dagger(\mathbb{Q}_{q,t}1) = 0.$$

Let us sketch another approach to (i) based on Theorem 2.4. We try to collect  $\alpha_l, \dots, \widetilde{\beta}', \dots, \widetilde{\alpha}'$  together in  $\lambda(s_i\pi_d)$ . Generally, it is possible only for admissible triples; however, here  $\alpha_l$  is simple non-affine and therefore can be moved using the Coxeter relations in  $\lambda(s_i\pi_d)$  to the first position. At a certain moment it will become next to  $\widetilde{\beta}'$  and we can apply Proposition 2.2,(iii). Then we can change the order of intertwiners corresponding to  $\{\alpha_l, \widetilde{\beta}', \widetilde{\alpha}'\}$  to the opposite, corresponding to  $\{\widetilde{\alpha}', \widetilde{\beta}', \alpha_l\}$ , thanks to Theorem 5.2,(c).

Due to the appearance of  $\alpha_l$ , the total chain of intertwiners for  $s_i\pi_d$  will result in zero for  $\{\widetilde{\alpha}', \widetilde{\beta}', \alpha_l\}$  since the corresponding chain of partial reduced decompositions will leave the set  $\pi_B$  after  $\alpha_l$ . The difference of

two expressions for  $\tilde{\Psi}_{s_i \pi_d}^\dagger(1)$  corresponding to the orderings  $\{\alpha_l, \tilde{\beta}', \tilde{\alpha}'\}$  and  $\{\tilde{\alpha}', \tilde{\beta}', \alpha_l\}$  is  $\text{const } \tilde{E}_c$  for  $\text{const} \neq 0$  (we use that  $t_i \neq 1$ ).

Concerning (ii), the orthogonality  $(\tilde{\beta}^v, \tilde{\beta}^{v'}) = 0$  makes it possible to proceed by induction. Compare with Corollary 10.7.  $\square$

Recall that the space  $\mathcal{V}(-c_\#)^\infty$  is linearly generated by the polynomials  $\tilde{E}_b$  for  $b$  such that  $q^{b_\#} = q^{c_\#}$ . They are defined for reduced chains have nonzero leading terms and are normalized as follows:

$$(12.35) \quad \tilde{E}_b - X_b \in \oplus_{a \succ b} \mathbb{Q}(q, t) X_a.$$

The  $\tilde{E}_b$  here is a  $Y$ -eigenvector if  $\tilde{E}_{b'}^\dagger = 0$  for all  $b' = \hat{w}'((0))$ , where  $\dagger$  indicate that the *standard decompositions* of  $\hat{w}' \in \tilde{\mathcal{B}}_o^0(\pi_b)$  are taken, possibly non-reduced.

Also recall that any  $\mathcal{V}(-c_\#)^\infty$  contains a unique  $Y$ -eigenvector up to proportionality if  $\text{Rad} = \{0\}$ ; indeed, otherwise a proper linear combination of two different eigenvectors from this space would belong to the radical. This eigenvector must be in the form  $E_b$  for (a unique)  $\pi_b <_0 \pi_c$ , i.e., for  $\pi_b$  obtained from a reduced decomposition of  $\pi_c$  by crossing out some singular simple reflections.

For *primary*  $b = b^\circ$ , the space  $\mathcal{V}(-b_\#)^\infty$  is one-dimensional and  $\tilde{E}_b = E_b$  is its generator. Otherwise there would exist  $a \succ b$  with the same weight, which contradicts the definition of primary elements. Recall the definition of the primary elements:  $q^{-b_\#^\circ} = q^{-b_\#}$  and  $q^{-b_\#^\circ} \neq q^{-a_\#}$  for  $a \succ b^\circ$ .

Due to the condition  $\text{Rad} = 0$ , we have the following implications for any  $\pi_c$  (all claims hold true for primary elements):

- (a) {reduced decompositions of  $\pi_c$  contain no singular reflections}  $\implies$
- (b) {no intertwiners  $\tau_+(T_j) - t^{1/2}$  appear in the corresponding  $\Psi_{\pi_c}$ }  $\implies$
- (c) {  $\dim \mathcal{V}_b = 1$  and  $\tilde{E}_b = E_b$  }

More generally, if  $\Psi_i$  is not infinity and not in the form  $\tau_+(T_j) - t^{1/2}$  the map

$$(12.36) \quad \mathcal{V}(-c_\#)^\infty \rightarrow \mathcal{V}(-b_\#)^\infty \quad \text{for reduced } \pi_b = s_i \pi_c$$

is *injective*. It suffices to check that it does not kill  $Y$ -eigenvectors. This results from the following lemma, which is a simple application of Corollary 6.6.

**Lemma 12.6.** *Let  $\Psi_i^c \neq \infty$ . We do not assume that  $l(s_i \pi_c) = 1 + l(\pi_c)$ .*

(i) The polynomial  $E' = \Psi_i(E)$  belongs to  $\text{Rad}$  for a  $Y$ -eigenvector  $E \notin \text{Rad}$  of weight  $-c_\sharp$  if and only if  $\Psi_i^c$  is proportional to  $\tau_+(T_i) - t_i^{1/2}$ , equivalently, (see (11.10)),

$$q_\alpha^{k_\alpha + (\tilde{\alpha}^\vee, c_- + d) - (\alpha^\vee, \rho_k)} = 1 \quad \text{for } \tilde{\alpha} = u_c(\alpha_i).$$

(ii) Provided that  $t_\nu \neq -1$  for all  $\nu$ , the intertwiner

$$\Psi_i : V'_c \rightarrow V'_b, \quad \text{where } V'_c = \tilde{V}_c \text{ modulo } \text{Rad},$$

is injective if  $\Psi_i^c$  is proportional to  $\tau_+(T_i) + t_i^{-1/2}$  for  $i \geq 0$ .  $\square$

Note that the demonstration of (i,ii) is immediate if  $i > 0$ . Say, if  $\Psi_i^c = T_i + t_i^{-1/2}$ , then  $\{E', 1\} = (t_i^{1/2} + t_i^{-1/2})\{E, 1\} \neq 0$ , so obviously  $E' \notin \text{Rad}$ .

Let us assume that there exists  $a \in B$  such that  $\tilde{R}^{-1} \subset \lambda(\pi_a)$  and  $\tilde{R}^{0,1} \cap \lambda(\pi_a) = \emptyset$ . It is always the case for the simply-laced root systems; use Proposition 1.4 and follow (12.32) for  $b_0, b_1$  proportional to  $\rho$ . Then Lemma 12.6 gives that an arbitrary  $\mathcal{H}^b$ -submodule  $\mathcal{V}'$  of  $\mathcal{V}$  subject to  $\text{Rad} = \{0\}$  contains infinitely many eigenvectors  $E_b$  such that *no singular intertwiners appear in the reduced chains for  $E_b$* , i.e.,  $\tilde{R}^{0,1} \cap \lambda(\pi_b) = \emptyset$ . We do not need this claim too much but it simplifies the considerations below.

**12.5. Main Theorem.** We are now in position to formulate and prove the main theorem of this part of the paper; the notations from the previous section are used.

**Main Theorem 12.7.** (i) Imposing (11.14), let the radical  $\text{Rad}$  of the form  $\{, \}$  be zero. Then the polynomial representation  $\mathcal{V}$  can be reducible only in the following cases:

$$(12.37) \quad \tilde{B}_n : k_{\text{lng}} = -\frac{s}{2g}, \quad \text{provided that } (s, 2g) = 1,$$

$$\text{where } n > g > n/2, \quad g \in 1 + \mathbb{Z}_+ \ni s,$$

$$\tilde{G}_2 : k_{\text{lng}} = -s/3, \quad \text{as } s \in 1 + \mathbb{Z}_+, \quad (s, 3) = 1,$$

$$\tilde{F}_4 : 4k_{\text{lng}} \in -1 - 2\mathbb{Z}_+ \quad \text{or} \quad 6k_{\text{lng}} + 2k_{\text{sht}} \in -1 - 2\mathbb{Z}_+.$$

If  $k_{\text{sht}}$  is generic in this list, then  $\text{Rad} = \{0\}$  and, indeed,  $\mathcal{V}$  is a reducible  $Y$ -semisimple  $\mathcal{H}^b$ -module; it remains reducible for any  $k_{\text{sht}}$ , however the radical may become nonzero.

(ii) Without imposing (11.14), the the above conditions from (i) become:

$$(12.38) \quad \begin{aligned} \tilde{B}_n : t_{\text{lng}}^{-g} &\in q^{1+\mathbb{Z}_+} \setminus q^{2+2\mathbb{Z}_+}, \quad \tilde{G}_2 : t_{\text{lng}}^{-1} \in q^{1+\mathbb{Z}_+} \setminus q^{3+3\mathbb{Z}_+}, \\ \tilde{F}_4 : t_{\text{lng}}^{-2} &\in q^{1+\mathbb{Z}_+} \setminus q^{2+2\mathbb{Z}_+} \quad \text{or} \quad t_{\text{lng}}^{-3} t_{\text{sht}}^{-2} \in q^{1+\mathbb{Z}_+} \setminus q^{2+2\mathbb{Z}_+}, \end{aligned}$$

where  $n/2 < g < n$  and  $t_{\text{lng}}^{-g'} \notin q^{1+\mathbb{Z}_+}$  for any  $g'$  such that  $\mathbb{Z}_+ \ni g' < g$  in the case of  $\tilde{B}_n$ . If  $t_{\text{sht}}$  is generic in this list, then  $\text{Rad} = \{0\}$  and  $\mathcal{V}$  is a reducible  $Y$ -semisimple  $\mathcal{H}^b$ -module.

(iii) Continuing (ii), the remaining cases when  $\text{Rad}$  is  $\{0\}$  but  $\mathcal{V}$  may be reducible are those from Lemma 12.2, (ii):

$$(12.39) \quad \begin{aligned} \tilde{C}_n : t_{\text{sht}}^{-g} &\in -q^{\mathbb{Z}_+}, \quad \text{where } n > g > \frac{n}{2} \\ t_{\text{sht}}^{-g'} &\notin \pm q^{\mathbb{Z}_+} \quad \text{for } \mathbb{Z}_+ \ni g' < g, \\ \tilde{G}_2 : t_{\text{sht}}^{-1} &\in \rho q^{\mathbb{Z}_+} \quad \text{for } \rho^2 + \rho + 1 = 0, \\ \tilde{F}_4 : t_{\text{sht}}^{-2} &\in -q^{\mathbb{Z}_+} \quad \text{or } t_{\text{lng}}^{-1} t_{\text{sht}}^{-3} \in -q^{\mathbb{Z}_+}. \end{aligned}$$

If  $t_{\text{lng}}$  is generic in this list, then  $\text{Rad} = \{0\}$  and  $\mathcal{V}$  is a reducible  $Y$ -semisimple  $\mathcal{H}^b$ -module.

*Proof.* We will assume that  $t_\nu \notin q^{-\nu(1+\mathbb{Z}_+)}$  for any  $\nu = \nu_{\text{lng}}, \nu_{\text{sht}}$  respectively for  $B, F, G$  (then  $\nu = \nu_{\text{lng}}$ ) and  $C, F, G$  (then  $\nu = \nu_{\text{sht}}$ ). Technically, we will need to exclude the case  $t \in q^{(1+\mathbb{Z}_+)}$  for the simply-laced  $R$  too; it will be considered separately.

We also need to consider separately the cases  $[-1220, 2j] \in \tilde{R}^0$  and  $[-0121, j] \in \tilde{R}^0$  for  $\tilde{F}_4$ . Without going into detail, the irreducibility of  $\mathcal{V}$  can be deduced directly from Key Lemma 12.5 in these two cases.

Thus, apart from these cases and those listed in (12.37) or, more generally, in (12.38, 12.39), let us suppose that  $\mathcal{V}$  has a proper  $\mathcal{H}^b$ -submodule  $\mathcal{V}'$ .

Using Corollary 12.4, we find the portion  $s_i \pi_c$  of the reduced decomposition of  $\pi_a$  constructed there such that  $E_{c^\circ} \notin \mathcal{V}'$  but  $E_{\bar{c}^\circ} \in \mathcal{V}'$  for  $\pi_{\bar{c}} = s_i \pi_c$ . Note that we examine *primary*  $c = c^\circ$  ( $E_{\bar{c}}$  may remain outside  $\mathcal{V}'$ ). We use that  $\mathcal{V}'$  contains all  $E_b$  with sufficiently large  $\lambda(\pi_b)$ .

Then we use Key Lemma 12.5, (i) and find  $\pi_b = \hat{w} s_i \pi_c$  as  $l(\hat{w}) + l(\pi_c) = l(\pi_b)$  and  $\pi_d = \hat{w}(i') s_i \pi_c$  for  $\hat{w}(i') \stackrel{\text{def}}{=} \hat{w}^{-1} s_{i'} \hat{w}$ . The corresponding intertwiner  $\Psi_{\hat{w}(i')}$  will be applied to  $E_{c^\circ}$ . The result can be either proportional to  $E_{c^\circ}$ , which contradicts to the assumption  $E_{c^\circ} \notin \mathcal{V}'$ , or

can be another (nonzero)  $Y$ -eigenvector for the same eigenvalue, which contradicts to  $Rad = \{0\}$ . It gives the required.

If claim (ii) of Key Lemma 12.5 is used here, then the combinatorial part (Corollary 12.4) can be reduced significantly. Namely, we do not need to use  $\pi_a$  with the *strict alternation*  $\tilde{R}^{-1} \rightarrow \tilde{R}^0$ ; weaker conditions are sufficient.

We note that Key Lemma 12.5 can be generalized in many ways; the following its variant is needed to manage the case  $k \in -1 - Z_+$ . Let  $k = -g \in -1 - Z_+$ , more precisely,  $t = q^{-g}$ ; for the sake of definiteness only the simply-laced case will be discussed below and we will assume that  $\rho \in B$ . Then  $\lambda(\pi_{a_+})$  for the element  $a = a_+ = g\rho \in B_+$  contains  $\tilde{R}_+^{-1}$  and also  $\lambda(\pi_{a_+}) \cap \tilde{R}_+^0 = \emptyset$ ; see (1.25). If  $a_- = -g\rho$  is taken, then

$$(12.40) \quad \tilde{R}_+^0 \subset \lambda(\pi_{a_-}) \setminus \lambda(\pi_{a_+}), \quad \lambda(\pi_{a_-}) \cap \tilde{R}_+^1 = \emptyset.$$

Using the chain of intertwiners for  $\pi_{a_-}$  followed by its reverse, where the singular intertwiners are omitted, we obtain a *nonzero* eigenvector with the same eigenvalue as for  $E_0 = 1$ . It is a variant of Key Lemma 12.5,(ii); the verification is similar to the considerations of Proposition 10.12. It gives the irreducibility of  $\mathcal{V}$  for  $t = q^{-g}$ , which is in a sense “the most non-semisimple” case.

Let us outline a justification of the theorem based directly on Lemma 12.3, without using a “global”  $\pi_a$  satisfying the *alternation property*.

*Step 1.* Similar to the above consideration, one can find  $E_{\bar{c}}$  in  $\mathcal{V}'$  for *primary*  $\bar{c} = \bar{c}^\circ$  such that  $\pi_{\bar{c}} = s_i \pi_c$  for *primary*  $c = c^\circ$  and the index  $i \geq 0$  satisfying (12.15) from Lemma 12.1 and with the corresponding  $E_c$  *not* in  $\mathcal{V}'$ . Indeed, let  $\bar{c}$  be *primary* such that  $\pi_{\bar{c}} = s_i \pi_c$  (the decomposition is reduced) for some  $c$  satisfying  $\tilde{E}_c \notin \mathcal{V}'$ .

We may assume that  $l(\pi_{\bar{c}})$  is minimal possible provided that  $\tilde{E}_c \notin \mathcal{V}'$ . If  $c$  is primary, then Lemma 12.6 gives that the intertwiner  $\tilde{\Psi}_i^c$  can be only of type  $(\tau_+(T_i) + t_i^{-1/2})$ .

If  $c$  is not primary then  $\tilde{E}_c$  is *not* a  $Y$ -eigenvector. Then  $\tilde{\Psi}_i^c$  is proportional to  $(\tau_+(T_i) - t_i^{1/2})$ . Taking  $\tilde{\Psi}_i^c$  is not sufficient here since

$$\tilde{\Psi}_i = \Psi_i = (\tau_+(T_i) + \frac{t_i^{1/2} - t_i^{-1/2}}{Y_{\alpha_i}^{-1} - 1})$$

involves  $Y_{\alpha_i}$  when applied to  $\tilde{E}_c$ . One has:

$$(\Psi_i)^2 = \frac{(t_i^{1/2}Y_{\alpha_i}^{-1} - t_i^{-1/2})(t_i^{1/2}Y_{\alpha_i} - t_i^{-1/2})}{(Y_{\alpha_i}^{-1} - 1)(Y_{\alpha_i} - 1)} \quad \text{and}$$

$$\tilde{\Psi}_i^2(\tilde{E}_c) = \tilde{\Psi}_i(E_{\bar{c}}) = \text{const } E_{c^\circ} \quad \text{for } \text{const} \neq 0.$$

Recall that  $E_{c^\circ}$  generates the space of  $Y$ -eigenvectors in  $\mathcal{V}_c$ . Therefore  $E_{c^\circ} \in \mathcal{V}'$ , which contradicts to the minimality of  $l(\pi_{\bar{c}})$ .

*Step 2.* Now let  $\tilde{\beta}', i', \pi_b$  be from Lemma 12.1 applied to  $c = c^\circ, i$ . We take a reduced decomposition of  $\pi_b$  extending that of  $\pi_c$ . We will also assume that  $l(\pi_b)$  is the minimal possible such that  $\tilde{E}_c \notin \mathcal{V}' \ni \tilde{E}_b$ . Then there are no *singular roots*  $\tilde{\beta}'' \in \lambda(\pi_b) \cap \tilde{R}_+^0[-]$  after  $\pi_c$  in  $\pi_b$  of the same type as  $\tilde{\beta}'$ ; it is straightforward to check that other singular roots (if any) in  $\pi_b$  can be removed by diminishing  $\pi_b$ .

Recall that applying  $\tilde{\Psi}$ -intertwiners along a given reduced decomposition of any element from  $\pi_B$  is always nonzero. Therefore,  $\Psi_{\hat{w}}(E_c) = E_b$  for  $\hat{w} = \pi_b \pi_c^{-1}$ , and  $b$  must be primary too due to  $Rad = \{0\}$ . Moreover, intertwiners of type  $(\tau_+(T) - t^{1/2})$  cannot appear in  $\Psi_{\hat{w}}$  because their (*nonzero*) images would belong to  $Rad$  (Lemma 12.6).

*Step 3.* The only situation that may prevent us from using Key Lemma 12.5 is the appearance of new  $\tilde{\alpha}'' \in \lambda(\pi_b) \cap \tilde{R}_+^1[-]$  associated to the same  $\tilde{\beta}'$  (the last root in  $\lambda(s_{i'}\pi_b)$ ) in the same way as  $\tilde{\alpha}'$ . Recall that any such  $\tilde{\alpha}''$  must appear *before*  $\tilde{\beta}'$  in any reduced decompositions of the elements from  $\pi_B$ .

Due to (12.25):

$$\tilde{\alpha}' = -\tilde{\beta}'(\alpha_l), \quad \tilde{\alpha}'' = -\tilde{\beta}'(\alpha_m), \quad \text{where } (\alpha_l, \alpha_m) = 0.$$

Applying Corollary 3.6, one can make  $\tilde{\alpha}''$  *before*  $\tilde{\alpha}'$  in  $\lambda(\pi_b)$  using the Coxeter transformations in this set. The procedure is as follows.

We can assume that  $\hat{v} = s_i \hat{w}^{-1} s_{i'} \hat{w} s_i \pi_c$  is reduced by diminishing  $\pi_c$  if necessary; here  $\pi_b = \hat{w} \pi_c$  (cf. Lemma 12.5). Since  $\alpha_l$  is from  $R_+$ , the triple  $\{\alpha_l, \tilde{\beta}', \tilde{\alpha}'\}$  can be made consecutive using the Coxeter transformations in the sequence  $\lambda(\hat{v})$ . Using Main Theorem 2.4, the set  $\lambda(\hat{v}) \cap R'_+$  must contain at least two simple roots from  $R'_+$  for an arbitrary subsystem  $R' \supset \{\alpha_l, \tilde{\beta}', \tilde{\alpha}'\}$  of type  $B_3, C_3$  or  $D_4$  in this theorem ( $\tilde{\beta}' = \epsilon_1 + \epsilon_2$  for  $B, C_3$  and  $\tilde{\beta}' = \theta^4$  for  $D_4$ ).



Using Coxeter transformations *inside*  $\pi_b$ , one makes these roots *after*  $\tilde{\alpha}'$ , changing the position of  $\tilde{\alpha}'$  if necessary. One of them (at least) must be orthogonal to  $\tilde{\beta}'$  and therefore will “disappear” in the corresponding  $\hat{v}$ , which contradicts to the minimality of  $l(\pi_b)$  imposed above.

We conclude that  $\{\alpha_i, \tilde{\beta}', \tilde{\alpha}'\}$  can be assumed satisfying Corollary 3.6,(iii). It gives that  $\tilde{\alpha}'$  can be transposed with  $\tilde{\alpha}''$  (or with all such  $\tilde{\alpha}''$  if there are several of them). Note that since  $\dim \mathcal{V}_c = 1$  and  $\dim \mathcal{V}_b = 1$ , applying Coxeter transformations will not change the corresponding  $E$ -polynomials (generally, it may influence the  $\tilde{E}$ -polynomials).

If  $E_c$  remains *not* in  $\mathcal{V}'$ , then it concludes Step 3. Otherwise we switch to  $\tilde{\alpha}''$  and proceed by induction using the *links* for creating a *zigzag* in  $\tilde{R}^0 \cup \tilde{R}^{-1}$  in the terminology of Lemma 12.3. Eventually, we will make this *zigzag* maximal. Then it will contain a *boundary* root  $\tilde{\beta}^* \in \tilde{R}_+^0[-]$  (with a unique link to  $\tilde{R}_+^{-1}$ ) and there will be no roots from  $\tilde{R}_+^1[-]$  between  $\tilde{\beta}^*$  and the previous  $\tilde{\alpha}^* \in \tilde{R}_+^{-1}[-]$ .

*Step 4.* In the absence of  $\tilde{\alpha}'' \in \lambda(\pi_b) \cap \tilde{R}_+^1[-]$  after  $\tilde{\alpha}'$ , one may apply Key Lemma 12.5,(i). Let us recall the construction. Setting  $\pi_b = s_{i_u} \cdots s_{i_1} \pi_c$  for  $i = i_1$  and  $\pi_{\tilde{b}} = s_{i'} \pi_c$ , we consider the elements  $\pi_d = s_{i_2} \cdots s_{i_u} s_{i'} \pi_b$  and  $\pi_{\tilde{d}} = s_i \pi_d$ . One has  $\pi_{\tilde{d}} = \pi_c s_{\tilde{\beta}'}$  with  $s_{\tilde{\beta}'} \in \tilde{W}^0$  (since  $\tilde{\alpha}' \in \tilde{R}^0$ ).

We do not claim that the product for  $\pi_{\tilde{d}}$  is a reduced decomposition but the reductions can be only with the reflections corresponding to invertible intertwiners. Such transformations are acceptable for this proof.

Here  $\pi_{\tilde{d}}$  belongs to  $\pi_B$  if the last root in  $\lambda(\pi_{\tilde{d}})$  does. If it is the case, we obtain a  $Y$ -eigenvector  $E_{\pi_{\tilde{d}}}$  non-proportional to  $E_c$  and with the same  $Y$ -eigenvalue; this contradicts to  $Rad = \{0\}$ . Otherwise, the last root in  $\lambda(\pi_{\tilde{d}})$  is in the form  $\alpha_l$  for  $l > 0$  and Key Lemma 12.5 shows that the corresponding chain of intertwiners results in  $E_c \in \mathcal{V}'$ , a contradiction. This contradiction concludes this step and ( a sketch of the) the proof irreducibility of  $\mathcal{V}$  without using Corollary 12.4. Recall that  $Rad = \{0\}$  and the exceptional cases listed in (12.38) and (12.39) were excluded.

We will check the claim about reducibility of  $\mathcal{V}$  when the  $t$ -parameters are from these lists (for generic  $t_{\text{sh}} or  $t_{\text{lng}}$ ) in the cases of  $\bar{B}, \bar{C}_n$ . The remaining three cases are very much similar.$

Let us impose (12.38) and (12.39) (see also (12.22) and (12.23)) and verify that  $\mathcal{V}$  is reducible (and semisimple) though  $Rad = \{0\}$ . This consideration extends the example of  $\tilde{B}_n$  considered in [C11].

*Example of  $\tilde{B}_n$ .* We assume that  $k_{\text{sht}}$  is generic. According to (7.22), the condition  $t_{\text{lng}}^{2m} \notin q^{-2-2\mathbb{Z}_+}$  for  $1 < m \leq n$  is necessary and sufficient for  $Rad = \{0\}$ . In this case,  $\tilde{R}_+^0[-] = \emptyset = \tilde{R}_+^1[-]$ ; therefore,  $\mathcal{V}$  is semisimple.

The set  $\tilde{R}_+^{-1}[-]$  is nonempty if and only if  $t_{\text{lng}}^g = q^{-s}$  for  $1 \leq g < n$ ,  $s \in 1 + \mathbb{Z}_+$ . Namely, the short root  $[-\epsilon_{n-g}, s]$  belongs to this set:  $((-\epsilon_{n-g})^\vee, \rho_k) + k_{\text{sht}} = -2gk_{\text{lng}}$  (see (7.22)). Setting  $t_{\text{lng}} = \exp(\frac{2\pi i j}{g})q^{-s/g}$  for  $s \in 1 + \mathbb{Z}_+$ ,  $j \in \mathbb{Z}_+$ , if  $Rad = \{0\}$  then:

$$(s, 2) = 1 \quad \text{and} \quad t_{\text{lng}}^{g'} \notin \pm q^{-1-\mathbb{Z}_+} \quad \text{for} \quad 0 < g' < n, \quad g' \neq g,$$

in particular,  $g > n/2$ .

These conditions give that  $Rad = \{0\}$ ,  $\mathcal{V}$  is semisimple and at least one non-invertible intertwiner exists, which imply the reducibility of  $\mathcal{V}$ . For instance, either the relation  $(j, g) = 1$  (any odd  $s > 0$ ) or the relation  $(s, 2g) = 1$  (any  $j \geq 0$ ) is sufficient as  $n > g > n/2$ . Generally,  $(2j, s, 2g) = 1$  as  $n > g > n/2$  is necessary and sufficient. The simplest example is  $n = 3, g = 2, s = 1, j = 0$ .

*Example of  $\tilde{C}_n$ .* We assume that  $k_{\text{lng}}$  is generic and present this case following the previous one to emphasize that they are dual to each other (under  $q \leftrightarrow q'$ -duality). According to (8.31), the condition  $t_{\text{sht}}^m \notin q^{-1-\mathbb{Z}_+}$  for  $1 < m \leq n$  is necessary and sufficient for  $Rad = \{0\}$ . In this case,  $\tilde{R}_+^0[-] = \emptyset = \tilde{R}_+^1[-]$  and  $\mathcal{V}$  is semisimple.

The set  $\tilde{R}_+^{-1}[-]$  is nonempty if and only if  $t_{\text{sht}}^{2g} = q^{-2s}$  for  $1 \leq g < n$ ,  $s \in \mathbb{Z}_+$ . Namely, the long root  $[-2\epsilon_{n-g}, 2s]$  belongs to this set:  $((-2\epsilon_{n-g})^\vee, \rho_k) + k_{\text{lng}} = -gk_{\text{sht}}$ . Setting  $t_{\text{sht}} = \exp(\frac{2\pi i j}{2g})q^{-s/g}$  for  $s \in \mathbb{Z}_+$ ,  $j \in 1 + \mathbb{Z}_+$ , if  $Rad = \{0\}$  then:

$$(j, 2) = 1 \quad \text{and} \quad t_{\text{sht}}^{g'} \notin \pm q^{-1-\mathbb{Z}_+} \quad \text{for} \quad 0 < g' < n, \quad g' \neq g,$$

in particular,  $g > n/2$  must hold.

For instance, either the relation  $(j, 2g) = 1$  (any  $s \geq 0$ ) or the relation  $(s, g) = 1$  (any odd  $j \geq 1$ ) is sufficient. Generally, the condition  $(j, 2s, 2g) = 1$  as  $n > g > n/2$  is necessary and sufficient for reducibility of  $\mathcal{V}$  when  $Rad = \{0\}$ ;  $\mathcal{V}$  is semisimple in this case (for generic  $k_{\text{lng}}$ ). The simplest example is  $n = 3, g = 2, s = 0, j = 1$ .

To recapitulate, let us recall that Main Theorem 2.4 and Corollary 3.6 are not needed if Corollary 12.4 is used. Moreover, claim (ii) of Key Lemma 12.5 can simplify the combinatorial part of the proof.  $\square$

**Comment.** (i) As a matter of fact, this proof can be used to manage a more general problem of the irreducibility of the quotient  $\mathcal{V}/Rad$ . We need a proper generalization of Corollary 12.4 and the Zigzag Lemma 12.3. Note that the roots from  $\tilde{R}_+^1[-]$  can now appear between  $\tilde{\alpha}'$  and  $\tilde{\beta}'$  connected by a *link*. The following observation is more or less sufficient to manage this problem. The roots from  $\tilde{R}_+^1[-]$  represented in the form  $s_{\tilde{\beta}'}(\alpha_l)$  cannot appear in these intervals; such roots may occur only *after*  $\tilde{\beta}'$  in any  $\lambda(\pi_b)$ . A natural expectation is that if  $\mathcal{V}/Rad$  is reducible then it has a  $Y$ -semisimple quotient, although we do not have any confirmations beyond the case  $Rad = \{0\}$ .

(ii) More generally, the same method is expected to help with checking the irreducibility of constituents of  $\mathcal{V}$ . In the simply-laced case, one can expect that a chain of intertwiner can “enter” submodules of  $\mathcal{V}$  only at the places where the simple intertwiners become of type  $\tau_+(T) - t^{1/2}$ . If the latter intertwiners can be avoided, generally, such chain can be expected to remain in the same constituent of  $\mathcal{V}$ . Here  $\pi_a$  must be chosen similar to that in Corollary 12.4) or a proper variant of Lemma 12.3 must be used. A natural step in this direction is a direct proof (without any reference to the localization functor) of the Kasatani conjecture; see [Ka],[En].  $\square$

An important motivation of the constructive methods we study in this paper is that, generally, intertwiners are helpful for the theory of *square integrable*, *tempered* and similar irreducible AHA-modules. The analytic properties of the intertwiners acting in a particular module are directly related to its analytic type. See, e.g., [O4, O5]; for instance, the so-called residual sets are directly related to  $\tilde{R}^{0,\pm 1}$  considered in our paper. Here an explicit description in terms of intertwiners is needed, similar to what we do.

Paper [MTa] and some other related papers indicate that the classification of the square integrable AHA-modules can be made sufficiently explicit for the classical root systems. The combinatorics involved and the methods from [MTa] employed are sophisticated (and the relation to [KL1] is far from simple). It seems important to develop the *non-semisimple* technique of intertwiners from this paper toward [O5] and

[MTa]. Hopefully, switching to the DAHA-level can be productive here, but there are no direct confirmations so far.

To conclude, we will touch upon the  $q \leftrightarrow q'$ -duality for  $\Pi_{\widehat{R}}$  in the notation from Theorem 8.2. Let  $\{t_{\text{lng}}^{l_r} t_{\text{sht}}^{s_r}, 1 \leq r \leq d\}$  be the set of the  $t$ -powers in the numerator of the corresponding product formula for the Poincaré polynomial  $\Pi_R$  from (7.13) in the case  $t_{\text{lng}} = t_{\text{sht}}$  and from (7.15-7.18). We set  $h_r = l_r + s_r$  and denote the corresponding sum in the *denominator* of  $\Pi_R$  by  $n_r$ . Thus  $n_r$  always divides  $h_r$  and  $d = n$ ,  $n_r = 1$ ,  $h_r = m_r + 1$  for all  $r$  as  $t_{\text{lng}} = t_{\text{sht}}$ . Let  $\tau_r \stackrel{\text{def}}{=} (t_{\text{lng}}^{l_r} t_{\text{sht}}^{s_r})^{1/h_r}$ .

Equivalently, the  $k$ -terms of the positive affine exponents of  $\Pi_{\widehat{R}}$  from (8.38-8.41) are  $\{l_r k_{\text{lng}} + s_r k_{\text{sht}} + j, 1 \leq r \leq d\}$ . Respectively, the negative affine exponents are divisors of the *non-rational* positive affine exponents with the corresponding ratios equal to  $h_r/n_r$ . The number  $d$  coincides with the number of negative affine exponents such that  $j = 0$  from Theorem 8.2.

We can now rewrite  $\Pi_{\widehat{R}}$  in a  $q \leftrightarrow \exp(2\pi i)$ -symmetric form as follows (cf. Theorem 11.1) :

$$(12.41) \quad \Pi_{\widehat{R}} = \prod_{r=1}^d \frac{\prod_{j,j'=0}^{h_r-1} (1 - q^{j/h_r} e^{2\pi i j'/h_r} \tau_r)}{\prod_{j,j'=0}^{n_r-1} (1 - q^{j/n_r} e^{2\pi i j'/n_r} \tau_r)},$$

which is an indication that the localization functor from [GGOR],[VV2] can be extended to the general  $q, t$ -theory.

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